EVOLUTION AND INTERNAL DYNAMICS OF QUASI-STATICALLY SHEARED GRANULAR FLOWS

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Granular materials are collections of particles whose sizes range from microns to meters. They can be packed as a static solid pile and can also move like fluid or gas. When driven at a sufficiently low rate under a compressive load, the grains flow in a quasi-static regime, in which they creep while maintaining simultaneous contacts with multiple neighbors.

We investigate experimentally a quasi-static flow of glass beads packed and sheared in an annular channel. The experiments utilize techniques of refractive-index-matched fluorescent imaging, particle tracking, and simultaneous measurements of volume and boundary shear force. Under long-term shearing, a crystallization transition accompanied by a step-wise decrease of packing volume and shear force can occur. This transition also alters the structure of the internal velocity field. Boundary conditions can affect the crystalline ordering throughout the entire packing. We find that, even under identical boundary conditions and shearing, the evolution of the packing can lead to non-unique final states. The behavior in response to shearing is influenced by the past history of the packing.

Our measurements of the internal velocity fields have a dynamical range of five decades; parameters such as packing thickness and particle size are varied systematically. We demonstrate the impact of crystalline ordering on the spatial gradient of grain velocity - a significant change of local rheology occurs as a consequence of the coherent grain motion. Changing particle size does not influence the gradient of particle velocity significantly; the velocity decay length does not show a direct scaling with particle size. In addition, we
make time-resolved measurements of the anomalous mobility of internal grains immediately following the reversal of boundary motion; this transient behavior reflects the adjustment of the anisotropic contact network. We show that about 20 layers are needed to reveal bulk properties of a granular packing, such as the shear banding of velocity field, the development of distinct states of internal order, and the anomalous mobility upon shear reversal. The effects of stationary boundaries on the spatial decay of velocity with distance from the shearing surface are discussed, based on both theoretical analyses and experimental data on several systems.
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Chapter 1

Introduction
1.1 Slowly creeping granular materials

Granular materials are noteworthy both for the range of surprising phenomena they exhibit, and for the challenges they provide to theoretical explanation [4, 9, 18, 38]. Granular materials usually mean collections of particles, whose sizes range from microns to meters. They can be packed as a static solid pile, or can be driven to flow like fluid or gas. One extreme case of granular flows is the rapid-flow regime: the dynamics of rapidly colliding particles is often described using the framework of dissipative kinetic theory with flowing grains bearing some resemblance to molecules in a gas. On the other hand, when particles do not have sufficient relative speed to sustain the imposed pressure - either by an externally imposed compression or due to the weight of grains under gravity, these particles tend to collapse into a dense state of slow creeping. This regime of slowly creeping grains is often termed “quasi-static” in the sense that particles are almost always in static equilibrium, supported by direct contacts with multiple neighbors while creeping around each other. In this regime, the inertia of individual grains plays negligible role in the flow dynamics. Slow geological flows [31] provide examples in which grains throughout the entire granular packing are moving in the quasi-static regime. In granular flows that are highly inhomogeneous, quasi-static creeping and rapid flow can often coexist, such as the particles in industrial blenders [33, 39]. In Appendix A.1, we provide a more precise criterion and a dimensionless number to characterize the quasi-static regime.

Quasi-static granular flows can exhibit rich phenomena, such as long-term evolution and dependence on prior history. These complexities are also found in many other glassy or nearly jammed physical systems. A simple feature of granular particles is that they interact with each other via short-range direct contact forces, in contrast to the long range interactions often present for suspended colloids or other complex fluids. Experimentally, monitoring dense granular packings driven by continuous boundary shear is a useful way to study the evolution of quasi-static flows. For instance, an apparently steady state can gradually undergo substantial change or, in some cases, make a stepwise transition as we report in this work.

Experimentalists have investigated quasi-static granular flows from various points of view. Observations by Komatsu et al. [27] on particles in a thick but quasi 2D pile of
flowing sand imply that the velocity of creeping grains can extend far from the surface into the deep bulk, spanning several orders of magnitude. Behringer and co-workers [32] measure the local contact forces at the lower boundary of a slowly sheared dense granular packing: the fluctuation spectrum (with frequencies scaled by the boundary driving velocity) of local contact forces is invariant over a wide range of driving speeds. This observation illustrates the quasi-static nature of the creeping flow, i.e. the grains are almost always in static equilibrium. Mueth et al. [36] use an MRI technique to study particles in a sheared layer several grain diameters thick, and determine the internal mass flow with sub-grain-size spatial resolution. They have pointed out the effect of particle layering on the mass flow field. In addition, the fluctuating motion of granular particles at the bottom of a Coutte shear cell [35], or at its upper surface [5], has been measured; the relation between the local velocity gradient and the fluctuating components of grain motion has been studied in depth. More recently, an experimental study of slowly driven granular flows using a Coutte geometry modified with a split bottom [16] shows that the location of the shear band (the locus of local maximum of velocity gradient) is determined by the global geometry of a shear cell, while the width of the shear band can be influenced by the properties of the grains such as their size.

To develop understanding beyond a steady-state description of granular flows, it is important to observe time-dependent behaviors. For instance, it is found that when the amplitude of an uniformly imposed cyclic shearing is changed abruptly, grains inside a dense packing exhibit a stepwise descent (under the influence of gravity) [42]. In addition, signs of extraordinary grain motion in response to the reversal of non-uniform shearing have been observed in the surface flow of grains in a Coutte geometry [29]. (In Chapter 5, we analyze the response of a granular packing to the reversal of shearing, based on the measurements of internal grain motion and time-resolved volume measurements.) Theoretical works have demonstrated anisotropic texture developed in sheared granular packings [17, 40, 1]. Internal grain motions during the reversal of shearing presumably reflect the readjustment of the anisotropic texture.

Compared to the theory for rapid flows, the theory for quasi-static granular flows is less developed. A number of analytic theories and models involving frictional interaction and plasticity have been attempted [2, 12, 26, 34, 45, 59, 60], while some of the assumed
constitutive relations can be hard to verify on a microscopic level. For these slowly creeping granular particles, the contact forces between densely packed grains dominate the dynamics. Approaches using kinetic theory may be of limited value for these packed grains. For instance, it has been estimated [28] that for millimeter-sized glass beads moving at a relative speed of several bead diameters per second under the static pressure of a layer of material that is several centimeters thick, the mean kinetic energy is actually smaller than the average elastic energy, and may be the least important quantity in accounting for the creeping dynamics. Recent theories show that the elasticity of grains can often play a vital role in the rheology of dense granular flows [7]. It is also known that changing the stress level in a sheared large-scale granular packing can qualitatively alter its velocity profile as a result of differences in the contact network, even though the corresponding fractional deformations of individual grains are all very small [1].

The connection between the internal structure of the material and the rheology of flow is particularly interesting. Here we sample a few previous investigations on shear-induced ordering in systems not limited to granular flows. Shear-induced ordering has been observed in computer simulations of uniformly sheared particle suspensions, in which the fluid-mediated hydrodynamic interactions between particles govern the dynamics [47]—these computations do not impose a thermostat as one might do in other simulations intended for systems of smaller particles with prominent Brownian motions. In computer simulations of particles uniformly sheared under the influence of theoretical thermostats [15, 30, 49, 6, 13], intrinsic time scales controlled by the imposed temperature can be defined, such as the mean diffusion time or thermal collision interval. The observed ordering behavior often depends on the rate of the macroscopic shear. The simulations of sheared particles with repulsive interactions (mimicking the behavior of dilute charged colloidal particles [8]) suggest that the presence of a shear velocity field can either enhance or suppress particle ordering [49, 6]. However, due to the difficulties in connecting these computations to specific experimental conditions, the proper interpretation of the ordering in these simulations with artificial thermostats is apparently still controversial, as is pointed out in Ref. [13]. In experiments of uniformly sheared colloidal suspensions, researchers have demonstrated that particles can exhibit ordering that leads to significant change of rheology, e.g., in Ref. [41, 21].
Note that, for a thick sheared packing, the velocity of grains can extend many orders of magnitude. The characteristic time for local relative displacement between grains, defined by the inverse of the local velocity gradient, exhibits a wide spectrum. Some interesting questions arise: Can a sheared granular packing exhibit spatial order in the presence of such a strong inhomogeneity in space? How does the internal order impact the rheology of granular flow? Experimentalists have observed ordering of granular spheres when grains are poured into vibrating containers with suitable boundary conditions [43, 37]. Recent soft sphere simulations of a large-scale gravity-driven granular flow [48] also find intermittent crystallization on an ordered substrate. However, existing theories of densely packed granular flows [2, 12, 26, 34, 45, 59, 60] generally have not included the effect of spatial order, partly because of the scarcity of experimental information about the internal structure of steadily flowing granular particles.

1.2 Scope of the present work

The goal of the dissertation is to investigate the dynamics of dense granular flow in the quasi-static regime, with special focus on: (1) the time evolution of a shear flow, (2) the connection between internal structure and granular rheology, (3) the development of the bulk properties with the size of the packing, and (4) the factors that determines the spatial distribution of particle velocity. The model system contains glass particles in an annular channel. The granular material is subjected to a fixed vertical load, and is sheared from above at a constant or alternating speed. Simultaneous measurements of internal particle structure, granular volume, and boundary shear force are performed as the granular packing evolves over time. Internal velocity fields and time-resolved trajectories of particles are measured. This system provides information that is complimentary to prior experiments using standard Coutte cells, which usually feature a rotating cylinder and a free surface of grains.

We first focus on spherical particles of a nearly uniform size. More generally, polydispersity would introduce another factor affecting the evolution of the packing; a movie demonstrating the shear-induced segregation of a mixture of particles of different sizes in
our system is available on-line. In the present work, we limit the examples to mono-disperse spherical objects, whose evolution does not involve size segregation, and whose stationary state consequently has a spatially uniform composition. Partial results are published in Ref. [56] and in subsequent papers [53, 54, 52]. The extension to more realistic situations involving poly-disperity or non-spherical particulates is under way.

Also, we create flow conditions with a hierarchy of forces in which the typical contact force between grains is much larger than the weight of one grain, which is in turn much greater than the maximal drag force provided by interstitial fluid. Therefore, the presence of fluid for imaging purposes in this work does not play a significant role in the quasi-static motions of grains, except by reducing the friction between particles or between particles and stationary walls through lubrication.

1.3 Granular flows in other regimes

A number of papers on granular shear flows in the same geometry as in our work are reported in the literature, but with focus on other aspects and in different dynamical regimes. In addition to the local contact force measurements already cited [32], there are other experiments using low normal load and high shear rates. The relation between shear force and shear rate [19, 46], as well as the granular self diffusion as seen from the sidewall [25], are reported in these experiments focusing on the behaviors in the rapid-flow regime. In addition, some previous works in the same geometry focus on stick-slip behavior [11], whereas we impose a stationary driving in the precision measurement of the internal velocity fields.

Different dynamical regimes can coexist spatially in stationary flows. The high-speed shear flows cited as Refs. [19, 46, 25] are examples in a closed geometry. This spatial inhomogeneity is also commonly seen in experiments on granular flows with a free surface. For instance, in gravity-driven flows with an inclined grain surface [27, 50], distinct behaviors are found for the rapidly flowing particles near the upper surface and the slowly creeping grains deeper in the bulk. The kinematical inhomegeity of the standard Coutte flow can arise from the vertical gradient of compressive stress, which starts from zero at its free

\footnote{http://www.haverford.edu/physics-astro/gollub/internal_imaging}
surface and increases downwards. This inhomogeneity becomes increasingly important as
the upper regions are agitated by higher shear rates. This situation eventually becomes a
complicated 3D problem when the centrifugal force on individual grains are large enough.

The preceding examples do not cover the situation where the dynamical regimes of
a granular system alternate over time. The alternation can be induced by an external
periodic driving, or, it can be spontaneous in the circumstance that density waves occur
[55].
Chapter 2

Experimental Methods
Figure 2.1: Cross-section of the annular channel filled with glass beads, with the inset showing a schematic side view and the definition of coordinates being used in this work. The shear and normal load W are transmitted through a mono-layer of glass beads glued to the rotating aluminum-acrylic assembly that is free to move vertically while maintaining constant rotation rate. Three different bottom conditions are available: flat, mono-layer and bumpy. Volume change is determined by detecting the vertical displacement of the upper assembly. The torque required to maintain the rotation is measured by a force gauge coupled to the motor through a pulling wire (not shown). Both vertical and horizontal slices of internal fluorescent images can be captured by digital cameras.

### 2.1 Main apparatus

The main apparatus is an annular channel formed by two concentric stationary glass cylinders, which are smooth and transparent (Fig. 1). We typically use spherical soda lime glass beads (density $\approx 2.5$ g/cm$^3$, elastic modulus $\approx 63$ GPa, refractive index $\approx 1.54$, from Jaygo Inc.) of mean diameter $d = 0.68$ mm with a standard deviation 4% to fill the channel; the filling height is adjustable from a few particle diameters to $50d$. The distance between the inner and the outer walls is about $28.5d$ and the circumference is about $800d$. Fluid can be added to the inter-particle space when internal imaging (described below) is planned. The glass beads are driven by a rotating ring-shaped upper boundary to which a mono-layer of glass beads has been glued; the glued glass beads are packed at about the maximal area density and form local hexagonal structures with dislocations. Three
different bottom boundary conditions are available: (1) flat bottom - smooth acrylic; (2) mono-layer bottom (described above); and (3) bumpy bottom - a mixture of 0.6mm and 1.0mm glass beads glued at approximately equal area fraction. In addition, a transparent window (thick polycarbonate) is made at the bottom of one part of the channel. The top surface of the acrylic part of the rotating assembly is also polished, to provide an option for viewing from the top.

The upper boundary is attached to a precision bearing that allows free but strictly vertical movement while rotating. The weight of the upper boundary assembly $W$ provides a constant normal load of $(1.3\text{Kg})g$, which is about 12 times the total effective weight of the (fluid-immersed) grains at the typical filling height of $24d$. Extra precision is demanded during the assembly process to ensure that the two planes defined by the upper and lower boundaries are parallel, and that the gaps between the upper boundary and the cylindrical walls are smaller than $0.5d$ around the circumference at all times, while the upper boundary rotates. The small gaps, combined with a fluid reservoir at the upper edge of the inner wall, allow the interstitial gas or fluid to exit or re-enter the channel freely, while the granular particles cannot escape. This allows the determination of the granular volume change by monitoring the height of the upper assembly. In practice, the height is measured once per rotation by detecting the position of a horizontal metallic arm installed on the rotating assembly and extending beyond the outer radius of the channel. An inductive-capacitive sensor is used for this purpose. (The design of the extended arm enables us to avoid calibration errors when the sensor is displaced, e.g., when changing particles.)

The unidirectional rotation is driven by a stepping motor (DM-4001 by Aerotech Inc.) whose rotating shaft is rigidly connected to and aligned with the central shaft of our apparatus; the central shaft rotates with the same angular speed of the upper boundary, as shown on the figure. The combination of the micro-stepping of the motor and a 100X speed reduction gear box gives an angular resolution of $2.5 \times 10^{-5}$ rotations. Extra care is taken to maximize the rigidity of the driving mechanism. The deviation of motion of the upper boundary from an ideal constant-speed translation is less than $0.1d$; this is estimated by monitoring the projection (onto a distant screen) of a laser beam reflected from a mirror installed on the central shaft of our apparatus.

The shear force at the upper boundary can be determined if one measures the torque
transmitted from the motor to the upper boundary or, alternatively, the torque that is required to resist the motor’s tendency to rotate in the opposite direction. This is achieved by letting the motor stand freely on its own rotating shaft, decoupling it from its usual rigid support, attaching a pulling wire off center, and measuring the tension in the wire with a force gauge (PCB Piezotronics: KT-1102-01). However, the flexibility introduced by the pulling wire inevitably causes some unsteadiness in the upper boundary motion, so the motor is rigidly supported when the velocity profile is measured.

2.2 Internal imaging and image analysis

To image the interior, the inter-particle space is filled with a special hydrocarbon mixture (viscosity \( \approx 10 \text{cS} \), density \( \approx 1 \text{g/cc.} \), from Cargille Laboratories.). The refractive index of the fluid is fine-tuned to match that of the beads by adjusting the relative concentration of the ingredients to the accuracy of \( 10^{-3} \). The optimal index-matching condition is determined by minimizing the scattering near the forward direction while a green He-Ne laser beam (543.5nm) is sent though a test cell containing the mixture of beads and fluid. In gluing beads to the upper or lower boundary, adhesive (Norland NOA-68) with the closest refractive index to that of the glass beads is chosen and bubbles are avoided, in order to minimize the distortion of the image or the illuminating light sheet (described below) through these roughened boundaries.

A minimal amount of fluorescent dye (Exton Pyrromethene-580, fluorescent peak \( \approx 540 \text{nm} \)) is blended into the fluid. A thin laser sheet enters the imaging region through either the bottom window or the front wall, to activate the fluorescence of the dye and create an instantaneous 2D image slice \( I_{\text{slice}}(\vec{x},t) \) in which particles appear as shadows. The image can be captured by digital camera through either the front wall or the bottom window.

To create the laser sheet, a narrow beam from the 514.5nm line of an argon ion laser (Lexel-75) is first expanded isotropically (by a microscope objective), then collimated (by a convex lens), and finally compressed one-dimensionally by a set of cylindrical lenses. (For this purpose, we use a pair of convex and concave lenses to create a tunable effective focal length \( f \sim 2 \text{m} \), and a convex lens with \( f = 6 \text{mm} \), located between the focal point of the tunable lens pair and our imaging area.) The resulting laser sheet is narrower than
0.5d throughout the region of interest. The fluorescent images are filtered with a cut-off wave-length 550nm before reaching the camera, to screen out the undesired noise due to scattering of the excitation light.

To track the particles, an image is first converted to a 2D map $\mathcal{M}$ using a convolution

$$\mathcal{M}(\vec{x}, t) = \int G'(||\vec{x} - \vec{x}'||) \cdot I_{\text{slice}}(\vec{x}', t) \cdot d^2\vec{x}'$$

(2.1)

where $G'(r)$ stands for the derivative of a gaussian ring

$$G'(r) = \frac{d}{dr} e^{- (r - a_0)^2 / 2\sigma^2_0}$$

(2.2)

with radius $a_0$ being approximately the apparent radius of a particle in the image (from 9 to a few tens of pixels, depending on the image magnification), and the width $\sigma$ being one or a few pixels. In practice, the convolution takes advantage of FFT techniques for the sake of computational efficiency [44]. The bright peaks in each map $\mathcal{M}$ represent the centers of particles in that frame, and are found using previously developed computer routines [10] which also reconstruct particle trajectories from a set of sequential images. This convolution method is based on the idea of edge recognition (similar to Hough transformation [3]), and therefore can still resolve particles in the presence of certain degree of particle overlap (due to the finite thickness of the light sheet). When high precision measurement of instantaneous individual particle positions (as opposed to coarse-grained mean velocity measurement) is needed, the light sheet is set to be as narrow as possible, and a high threshold of the peak intensity is selected so that only very sharp peaks (contributed by particles whose centers are precisely on the symmetrical plane of the light sheet) are admitted as valid data. The ultimate limit of precision in determining particle positions is set by the pixel width, corresponding to a few tenths of a particle diameter (depending on magnification.) All computations are performed using IDL programming. Images demonstrating the quality of the particle recognition are included in Appendix B.1.1.

We also analyze the spatial spectrum of each image slice. For instance, the horizontal spatial periodicity of a vertical image slice $I_{\text{slice}}(x, z, t)$ can be represented by the following normalized Fourier spectrum averaged over depth $z$

$$f(k_x, t) = \langle \frac{|F(k_x, z, t)|}{(\int |F(k_x, z, t)|^2 dk_x)^{1/2}} \rangle_z$$

(2.3)
in which $F(k_x, z, t) = \mathcal{F}_x[I_{\text{slice}}(x, z, t)]$ stands for the Fourier transform in the $x$ direction. The denominator is to correct the variation of image intensity over $z$ and $t$. The intensity of a peak of this normalized spectrum $f(k_x, t)$ at a particular wave number $k_x^{(1)}$, corrected by subtraction of the background value, is defined as

$$I^{(1)}(t) = \frac{1}{2\epsilon} \int_{k_x^{(1)}-\epsilon}^{k_x^{(1)}+\epsilon} |f(k_x, t)|^2 dk_x - I^{(1)}_B(t)$$

where $I^{(1)}_B(t)$ is a background value determined by averaging $|f|^2$ over a broader spectral interval centered at $k^{(1)}$ but excluding the wave-numbers within $(k_x^{(1)} \pm \epsilon)$. When the wave-number $k_x^{(1)}$ is chosen to match the mean horizontal spacing ($\approx 1d$) of particle centers in the crystallized state, the calculated peak intensity $I^{(1)}(t)$ can be used as an indicator of the growth of the crystalline order.
Chapter 3

Crystallization Transition
The constant-speed rotation of the upper boundary provides a long-term shearing to the packed grains in the annular channel. For a wide range of conditions, the long-term shearing can induce a stepwise change in the internal structure, accompanied by a significant decrease of granular volume and boundary shear force. In this chapter, we characterize this process and its rheological consequences in detail, and describe the factors that influence the transition. The focus of this chapter is the shear-induced evolution from a disordered state to a crystallized state; a detailed comparison between the two states in terms of their 3D internal structures and velocity fields is provided in Chapter 5.

3.1 Characterization of the transition

In the experiments described in this and the next section, the annular channel contains 200g of glass beads (diameter \(d = 0.68\text{mm}\)), which can create a 24-layer structure when the crystallization occurs. Fig. 3.1 shows the simultaneous measurement of internal ordering, granular volume, shear force, and particle speed as functions of time. The disordered initial state is generally prepared by stirring the whole volume thoroughly. Time \(t = 0\) denotes the instant when the motion of the upper boundary begins to move at a speed of 12 particle diameters per second (12\(d/s\)); the bottom is flat.

In Fig. 3.1(a), the images show two instantaneous vertical slices at about one-third of the channel width from the outer vertical wall. They illustrate the disordered initial state \((t = 0)\) and the crystallized state \((t = 60000s)\). A sample of the instantaneous spatial Fourier spectrum \(f(k_x)\), defined in Section 2.2, is shown in the inset. The intensity of its primary peak \(I^{(1)}\) quantifies the degree of spatial order as a function of time. The displayed curve is smoothed over 20 frames (20 minutes in time) to reduce noise. The step rise of \(I^{(1)}\) marks the occurrence of crystallization.

The change of the total granular volume is determined by measuring the vertical displacement of the upper boundary \(h(t) - h(0)\) at fixed load, as shown in Fig. 3.1(b). The displacement, normalized by the mean height of the glass beads \(H_0\), reveals a fractional change of total volume of about 3 percent at the crystallization transition. The absolute total volume can be computed from the geometrical parameters of the channel with an uncertainty primarily due to the rough boundaries, and the packing fraction of the final
Figure 3.1: Simultaneous measurements of internal ordering, volume change, shear force, and particle speed, as functions of time. Glass beads are driven at a speed of $12d$ per second from above. (a) Vertical image slices before and after the crystallization, with $I^{(1)}$ showing the intensity of the primary peak of the FFT spectrum, as are defined in Section 2.2. (b) Fractional change of total height, as an indicator of total volume change. (c) Shear force, averaged once every rotation. (d) Particle speed averaged over the lower region $\Gamma$ of the cell, as marked on the upper right image. The speed is normalized by the velocity of the driving boundary $U_B$. 

\[ \mu \cdot \Gamma \]
Figure 3.2: Sample histories of evolution as the upper boundary translates at constant speeds starting at $t = 0$. The cell contains 200g of glassbeads with a flat bottom installed. In all cases, granular packings have been thoroughly stirred before $t = 0$ but have not undergone boundary shearing. The changes of height of the upper boundary $h(t)$ and shear force $\tau(t)$ are strongly correlated within each experiment. In cases (a,b) and in Figure 2, the glass beads are driven at exactly the same speed ($12d/s$), while the transition time exhibits stochastic variation. Case (c) is driven at a lower speed ($1.2d/s$). In all cases, the accumulated boundary translation at the time of the transition step is about $10^5d$ to $10^6d$ in the direction of the flow.
state is estimated as $(63 \pm 3)\%$.

Meanwhile, the shear force as a function of time shows a simultaneous step decrease of about 15 percent at the crystallization transition, as shown in Fig. 3.1(c). By adding extra weight to the upper boundary, we have also verified that the magnitude of the shear force is proportional to the vertical load imposed on the grains. When normalized by the vertical load $W$ (as shown by the right axis of Fig. 3.1c ), the shear force data gives an effective friction coefficient that goes from about 0.32 at the disordered state, to 0.27 at the crystallized state.

Fig. 3.1(d) shows the local average of grain velocities in a region far below the sheared interface as a function of time. This is computed by averaging the velocities of particles in the region $\Gamma$ indicated in the upper-right sample image, smoothed over 80 frames in time. The particles far below the sheared surface translate significantly more slowly after crystallization occurs, because the crystallized state has a stronger vertical decay of velocity (see Section 5.1.1.)

For a given driving speed and boundary condition, the typical time required for a disordered initial state to reach the crystallization transition varies stochastically within about an order of magnitude, as shown by the records displayed in Fig. 2 and Fig. 3.2(a,b). The period is roughly consistent with the time required for the particles near the bottom to move a distance of several particle diameters relative to their neighbors or the bottom. (This characteristic time of particle displacement can be either deduced from the velocity profiles provided in Chapter 5 or estimated from the values in Fig. 3.1d.) The length of the precursor corresponds to about $10^5d$ to $10^6d$ of accumulated translation of the upper boundary, where $d$ is the particle diameter. The experiment shown as Fig. 3.2(c) is driven at 1/10 of the typical driving speed, exhibiting the same features of transition with a longer precursor in terms of time but corresponding to the same order of accumulated translation of the driving boundary.

Despite of the stochastic nature of the transition time, we note in Figs. 3.1-3.2 that the records of the total volume and shear force are strongly correlated in each experiment. The similarity is not limited to the time of the major transition step, but also includes many sub-features of the curves. (The occasional spikes are produced by brief isolated jamming events when smaller particles get caught in the gaps between the moving upper boundary.
and the stationary sidewalls.) These two boundary measurements, volume change and shear force, can both serve as sensitive indicators of the state of evolution. Figs. 3.1-3.2 also show that a smooth initial dilation of granular volume can be detected within the first $10^4d$ of translation of the driving boundary; the exact amount of this dilation depends on the preparation of the initial state.

We have verified that the crystallization does not require a uniform speed of boundary translation, and is generic with or without interstitial fluid. In the absence of fluid, we find that the compaction steps also occur, but the accumulated translation of the driving boundary can be an order of magnitude longer than that for fluid-immersed particles under the same driving and boundary conditions. (See Section 3.3)

### 3.2 Role of bottom condition

There are three different bottom boundary conditions that can be used for our experiments: (1) flat bottom, (2) mono-layer bottom, and (3) bumpy bottom, as defined in Section 2.1. The results show that the tendency to crystallize has a non-monotonic dependence on the roughness of the bottom boundary. When we load the cell with 200g of initially disordered glass beads and drive the upper boundary at the same speed (12$d$/s), a spontaneous transition to a 24-layer crystallized flow occurs when using either the flat or the mono-layer bottom boundary (but not the bumpy boundary). Notably, experiments using the mono-layer bottom [see the data presented in Ref.[56]] typically require less translation of the upper boundary to trigger the crystallization transition than those using the flat bottom. Furthermore, when less material is used, for instance 100g of glass beads, a persistent 12-layer crystalline structure is quickly produced in experiments using the mono-layer bottom boundary, while the flat-bottom experiment can only induce incomplete ordering which appears to be intermittent in the observation window. These observations demonstrate that the crystallization is produced most easily and completely using the mono-layer boundary. The crystalline structure of the boundary, even if imperfect, appears to favor ordering in the bulk of the material, as occurs during epitaxial growth of ordinary solids.

In the case of the bumpy bottom, on the other hand, driving initially disordered glass beads for a long time gradually compacts the packing into a final state whose interior
remains disordered indefinitely. Interestingly, the crystallization is then dependent on the history prior to the onset of the long-term unidirectional shearing, as we discuss in detail in Chapter 4. In all cases, the asymptotic values of granular volume consistently indicate the final state of internal order.

Although the bottom boundary condition has a global effect on the crystallization of the granular packing, the roughness of the bottom boundary alone does not affect the velocity field in the deep bulk measurably, once the same state of internal structure is created. (See Section 5.1.1.) Furthermore, regardless of the bottom boundary conditions, the asymptotic value of shear force for the disordered or crystallized final state is, respectively, equal to the typical value before or after the transitional step shown in Fig. 3.1(c). In other words, the mean shear force is sensitive to the internal packing structure, but not to the roughness of the bottom boundary.

3.3 Timescales of transition and behaviors of packings of different thickness

We find that packings of different thickness require drastically different amounts of accumulated boundary translation (and subsequently the timescales) for the transition to occur. Using the mono-layer bottom, which is the most favorable condition for complete crystallization, we compare the compaction histories of sheared flows for different amounts of glass beads in the channel, as shown in Fig. 3.3(a) (also as in Ref. [56]); the data show that the characteristic time for transition is diminished by two orders of magnitude when the total thickness of the packing is halved, even though the same driving speed is used. In addition, the experiment is also performed in the absence of interstitial fluid, shown as Fig. 3.3(b). We find that the transition also occurs without fluid. However, for the same thickness and driving speed, the precursory period of the transition for a dry granular packing can be longer by an order of magnitude (days in this case, or $10^6d$ in terms of boundary displacement); the transition in the dry case can occur by means of multiple compaction steps, possibly involving material at different depths. This suggests that friction plays a significant role in the underlying dynamics, and that this frictional interaction can be altered by the lubrication provided by the interstitial fluid.
Figure 3.3: (a) Compaction histories for fluid-immersed grains of different layer thickness, as indicated by the total mass of granular material (and the number of layers in the final state). The curves are shifted vertically for clarity. The transition delay grows rapidly with depth. (b) Compaction history of dry grains. In all cases, the boundary speed is $12d/s$.

Fig. 3.4 demonstrates that the final thickness of the flow has a ‘quantized’ dependence on the total mass $M$ of the glass beads in the channel, corresponding to the creation of additional layers. This non-uniform increase of final thickness as $M$ is increased can also be observed in thicker samples up to at least 24 layers thick. On the other hand, a smooth linear increase (not shown) is observed for comparable experiments using the bumpy bottom, for which the glass beads do not crystallize.

It should be noted that the thinner flows exhibit incomplete crystallization at the transitional bands between adjacent zones, marked in Fig. 3.4 as the narrow shaded bands equivalent to about 1g of total fill. As the total packing thickness is reduced to about 6 layers (50g), these transitional bands can become as wide as 4g, which is about half of the
Figure 3.4: Non-smooth dependence of the final height $h_f(M)$ of the crystallized flows on the total amount (mass $M$) of glass beads. The height of the 100g case is used as a reference. The mono-layer boundary condition at the bottom is shared by all cases shown here. Specific numbers of well-defined ordered layers occur in discontinuous bands of $M$. The transition zones, in which crystallization is incomplete, are marked as shaded stripes.
mass accounting for each layer on average. The wider the transitional bands, the easier it is to miss the crystallization phenomenon. This may explain why crystallization was not emphasized in previous experiments using thin layer flows to investigate shear band dynamics. The width of the transitional bands becomes undetectable in flows as thick as 24 layers, for which we find the transitional band to be less than 1/8 of the mass of one layer.

On the other hand, extending the overall thickness of the packing indefinitely does not guarantee a uniform crystallization. When the thickness is increased to 30 layers or above, depending on the preparation procedure, a non-uniform state with its degree of local crystalline order gradually varying with height can be created, and the non-uniform state can persist as a final state in response to the long-term unidirectional shearing provided by the upper boundary. The influence of preparation history on the evolution of a sheared packing is described in Chapter 4, with the velocity fields of different stationary states being investigated in detail in Chapter 5.

3.4 Discussion

We have identified a simultaneous step decrease of volume, shear force, and grain speed (in the lower part of the sample volume) at the crystallization transition of the packed grains (Fig. 3.1). This observation indicates that the crystallized state transfers horizontal momentum downward less efficiently than does the disordered state, even though the crystallized state is denser. This observation demonstrates that granular density alone does not determine the rheology of the material. Detailed descriptions of the internal dynamics are provided in Chapter 5.

The observation that transition steps occur for both fluid-immersed and dry particles (Fig. 3.3) suggests that the shear-induced crystallization is generic with or without fluid; this is also consistent with the theoretical estimate that the particle-fluid interaction is negligible compared to the large contact forces between grains. However, the longer precursory period and the non-single transition steps in the dry-particle experiment indicate that the lubrication by the fluid may have changed the internal velocity field significantly. This example illustrates the importance of including inter-particle and particle-wall friction
in modeling the underlying dynamics of a densely packed granular flow.

A bumpy bottom boundary is sufficient to suppress the shear-induced crystallization everywhere in the deep random packing equivalent to 24 layers of grains. A mono-layer bottom is found to be the preferred boundary condition to complete the crystallization process. This fact may be attributed to the spatial periodicity of the mono-layer packing. A scenario similar to epitaxial growth [23, 37] with the lower boundary playing the role of template seems plausible in view of the sensitivity to bottom conditions (Section 3.2), the typical length of the precursory period (Fig. 3.1), the stochasticity of the required time (Fig. 3.2), and its drastic change with mean layer thickness (Fig. 3.3a). However, the degree of order visible in a particular slice of the whole annular channel appears not only inhomogeneous but also intermittent during the transition, because the ordering process is non-uniform around the channel. There is no clear evidence that the crystallization grows from the bottom upwards.

Our mean packing fraction in the crystallized state (63±3)% is lower than the theoretical value for a hexagonal close packing ($\pi/\sqrt{18} \approx 74\%$), because the extra space and defects are necessary to allow the flow and to accommodate the circular finite-sized container. On the other hand, the mean density of the crystallized state here is close to but slightly higher than $\pi/\sqrt{27} \approx 60\%$, which is the theoretical value for parallel planes of hexagonally packed particles with a spacing exactly equal to the particle diameter. Presumably the actual value is affected by the degree of overlaps or interpenetration of adjacent layers, and the density of defects. The packing fraction of the disordered initial state is roughly 3% less than that of the crystallized state, and is significantly lower than the reported values for random close packing ($\approx 64\%$) [51]; in our experiments, even the crystallized state is not denser than the conventional random close packing.
Chapter 4

Non-unique Selection of Final States
When using a sufficiently bumpy bottom boundary (described in Section 3.2), we find that the final state is not unique. Whether the interior of the dense packing evolves into a crystallized or a disordered final state as a consequence of the unidirectional driving of the mono-layer upper boundary depends on the prior history. In particular, applying a few initial cycles of oscillatory shear can significantly alter the evolution. In this chapter, we discuss the correlation between the prior history of a packing and the selection of its final state in response to a long-term steady shearing.

4.1 Stochastic outcomes of unidirectional shearing combined with oscillatory pre-treatment

In this section, we show that a few oscillatory cycles, if applied prior to a long-term unidirectional shearing, can induce crystallization in a situation where it would otherwise not occur. The protocol of the oscillatory cycle we use is to alternate the direction of shearing a few times: the boundary travels at a constant speed (typically $12d/s$) each way for about $10^2d$ before its direction is reversed. Further details regarding the internal motion during the reversal of shearing and the change of total volume associated with this reversal process are provided in Section 5.2.

When a few oscillatory cycles are applied just after the random packing is prepared by thoroughly stirring the entire volume, subsequently applying a sustained unidirectional shear can lead to either a crystallized final state or another final state that remains disordered in its interior. This is demonstrated by Fig. 4.1, where the vertical displacement of the upper boundary, or equivalently the change of total volume, is used as an indicator of the long-term evolution. During the oscillatory pre-treatment, the total volume exhibits successive compaction following each step of shear reversal, as shown by the gradual descend of the upper boundary in the stage of oscillatory shearing. The long-term evolution of the granular packing in response to the subsequent unidirectional shearing appears stochastic: the same number of oscillatory cycles can lead to different outcomes. Nevertheless, the general trend is that a large number of oscillatory cycles tends to produce a crystallized final state, while applying a unidirectional shearing alone with no oscillatory pre-treatment always leads to the disordered final state.
Figure 4.1: Non-unique selection of final states, indicated by the long-term evolution of the packing height $h(t)$. With only unidirectional shearing (labeled empty circles), the system always approaches a disordered final state (joining curves at the top right). On the other hand, applying a few cycles of oscillatory shearing to the uncompacted packing causes the system to evolve stochastically into either a disordered state, or a denser crystallized state.
Evidence from simultaneously recording the internal images shows that a large number of initial oscillatory cycles causes grains to organize into clearly identifiable layers in the upper half of the packing; this partial order tends to extend to the entire bulk as the packing evolves under a steady shearing. The spatial ordering in the field of view during the subsequent unidirectional shearing can appear intermittent; this observation suggests that the partial order induced by oscillatory shear in the upper region is not uniform around the annular channel.

4.2 Shear-induced stabilization of the disordered state

It is remarkable that, if the oscillatory cycles are applied after the material has experienced a long-term unidirectional shearing, then the oscillatory treatment becomes ineffective in producing internal order: even though oscillatory driving still causes the upper boundary to descend (in a way similar to the data in the case shown as Fig. 4.1 during the oscillatory shearing), simultaneous internal imaging reveals that the system shows no sign of partial ordering after as many as 60 reversals of boundary motion have been applied. The oscillatory boundary driving does not change the deep interior of a sufficiently compacted packing significantly. When the unidirectional shearing is resumed, the packing exhibits a gradual recovery to its previous stationary volume prior to the application of the oscillatory treatment. (See Section 5.2.2.)

On the other hand, a crystallized state is always stable against unsteady boundary shearing; the ordered structure can only be destroyed by stirring the packing manually.

4.3 Discussion

We have found that, for given driving and boundary conditions (a bumpy bottom), the final state of a continuously driven flow can be non-unique. The selection of a crystallized or a disordered final state partly depends on the preparation history prior to the long-term shearing, and can appear stochastic if the material has not been sufficiently compacted by unidirectional shear. In comparison to unidirectional shearing, oscillatory driving is more effective in producing the partial order that generally favors the evolution towards
a crystallized final state. This effectiveness may be related to the extra mobility of the packing during the adjustment of its internal texture to adapt the new direction of shearing. Further information about this phenomenon are reported in Section 5.2.1. In addition, the quasi-hexagonal structure of the mono-layer driving boundary may in part account for creation of this partial ordering.

Evidence from time series of internal images suggests that the the partial ordering induced by the oscillatory pre-treatment is inhomogeneous. In the subsequent unidirectional shearing, whether the system selects the route of global crystallization or complete elimination of the crystalline order may depend on the extent and spatial distribution of this partial ordering. This is perhaps the reason why the outcomes of the subsequent shear-induced evolution of internal structure appears stochastic.

Both the crystallized state and disordered state can be stable against oscillatory boundary motion, after being sufficiently compacted by a unidirectional shear. On the other hand, we also recognize that the reverse of a crystallization transition never occurs unless we reset the system by stirring the packing manually. A tentative explanation is that the crystallized state is less dissipative and is therefore a ‘preferred’ state of flow compared to the disordered state.

Our observation that unidirectional shearing gradually stabilizes a disordered state demonstrates that substantial change in the properties of a granular packing can slowly occur as a result of a long-term evolution. This change is a type of smooth evolution that is different from the sharp distinctive change of internal structure exemplified by the crystallization process. It is important that any successful model for dense granular flow be able to account for both types of evolution under long-term shearing.

As a possible theoretical approach to understanding the phenomena described in this section, one might regard the crystallized and disordered states as attractors in a space-time dynamical system. The structure of the phase space and its attractors would be affected by variables such as the total volume and the boundary conditions. For example, for the flat or mono-layer bottom boundary, there is only a single attractor (the ordered state), while for the bumpy bottom, there are two attractors. The role of the oscillatory shearing could perhaps be modeled as a perturbation affecting the current position of the system point relative to the attractor(s). It would be interesting to consider how to define
the variables spanning phase space and the underlying differential equations to implement such a ‘dynamical systems’ explanation.

In this chapter, the discussion is limited to a simple case where there are only two distinct final states (crystallized and disordered states) and both states are uniform and stable. Here we have excluded the situation for thinner packings where the crystallization can be intermittent, as described in Section 3.3. The cases of thick packings (30 layers or more), where non-uniform states with a gradually varying degree of ordering can be created and be stabilized, are also beyond the scope of this chapter.
Chapter 5

Quasi-static Internal Dynamics
In this chapter, we focus on the dynamics of individual grains, based on precision measurements of the internal velocity field, and time-resolved trajectories of particles during both the steady states and transient responses. The dynamics is investigated in the quasi-static regime, in which particles are nearly in static equilibrium and maintain contacts with multiple neighbors. Theoretical justifications of the range of normal stress and driving rates for generating quasi-static flows are provided in Appendix A.1. In Section 5.1.1, the contrasting behavior of the disordered and crystallized states of the flows is analyzed in detail. In Section 5.1.2, we examine the roles of the particle size and the system size (the thickness of the packing) on the mean velocity fields. The rate-independence of the velocity field and the proportionality of the shear force to the normal load are described in Section 5.1.3 and 5.1.4. In Section 5.2, we analyze the internal motion captured during the reversal of shearing; investigations of this transient behavior provide insights into the rheology of a quasi-static granular bulk beyond the steady state. The effects of stationary boundaries on shear banding, comparisons of quasi-static granular flows in different geometries, the gradual development of bulk properties of a granular packing as its thickness increases, and the contrast between a sheared granular packing to ordinary fluids are discussed in Section 5.3; the discussions conclude with a heuristic model as an attempt to explain the observed shear banding.

Throughout this dissertation, we use the following conventions for the three mutually orthogonal coordinates: $x$ refers to the horizontal direction of the mean flow in the region of imaging; the $z$ axis points vertically upwards with $z = 0$ marking the height of the interface between the rotating boundary and the beads; the $y$ axis is perpendicular to both $x$ and $z$ and points towards the center of the annular channel, with $y = 0$ denoting the location of the outer wall.

## 5.1 Steady flows

### 5.1.1 Effects of crystalline order

Dramatic differences in both the ordering and the shape of the velocity field are found before and after the crystallization transition. Fig. 5.1 illustrates the three-dimensional features of these changes. In the disordered state shown in (a), the grain velocity exhibits a significant
decay towards the sidewalls. The ordered state (b) forms hexagonal sheets sliding against adjacent layers, and shows no significant decay of speed towards the sidewalls. Point defects and dislocations are visible from the sample image in case (b); the time-resolved image sequence (with movies available via the online archive associated with Ref. [56] show that point defects often persist as they move across the field of view.

Fig. 5.2 shows the time-resolved trajectories of particles sampled at horizontal internal planes. In each graph, coordinates of centers of all particles captured in a horizontal \((xy)\) region of size \(8d\) by \(8d\) with a depth of field of about \(1d\) (set by the thickness of the laser light sheet) are overlaid as functions of time. In a disordered state shown as graph (a), the grains show no obvious spatial correlations, and the trajectories here typically end within a net travel distance along \(y\) comparable to the grain diameter, presumably as a consequence of particles' leaving the depth of field. In contrast, Fig. 5.2(b) shows that grain motions in the crystallized state are strongly correlated across the entire field of view. Fig. 5.2(c) illustrates longitudinal motions of crystallized grains, showing the collective translation that is temporally non-uniform. The behavior of crystallized layers is consistent with the zig-zagging of hexagonal structures over adjacent layers, as shown in online movies associated with Ref. [56]. Similar zig-zag patterns have also been seen in experiments and simulations on other physical systems, such as suspended colloids [49, 41]. In the quasi-static flow investigated here, the typical patterns only depend on the state of order but not on the actual shear rate: for a given state of order, the behavior is similar regardless of the large differences in local time scales at different heights and with different driving speed. For instance, the characteristic periods of the zig-zagging in Fig. 5.2(b,c), shown by horizontal bars on the graph, consistently correspond to a few particle diameters of local translation, or to the local shear strains of the order of unity, because these patterns are a consequence of the relative motion of stacked hexagonal planar structures.

The measurement of the vertical profiles of mean velocity for different final states is presented in Fig. 5.3. For either the crystallized or the disordered state, the sampling starts after the packing has evolved under unidirectional shearing at a speed of \(12d/s\) for several days, corresponding to \(10^6d\) of total translation of the upper boundary. This ensures that the velocity field has reached the asymptotic state. Since the height-dependent grain velocity varies over several decades, we extract particle velocities from local imaging of
patches recorded at different frame rates (as illustrated in Appendix B); each data point is determined from a local average over $10^2d$ of accumulated particle displacement. The amount of stochastic variation of the mean value under this finite-length sampling is of the order of 30%, labeled as $\delta$ on the graph. Further information regarding measurement uncertainty is provided in Appendix B. For the disordered final state created with the same amount of glass beads, the height-dependent asymptotic velocity is measured at two vertical planes during one experiment. This is because, unlike the crystallized state, the grain velocity of the disordered state decays towards the sidewalls (Fig. 5.1). But the decay is generally not more than a factor of 10; there exists a finite slip velocity at the sidewalls.

Our measurements show that the downward decay of the asymptotic mean velocity in the crystallized state is significantly steeper than in the disordered state. The change of velocity profile due to crystalline order is also qualitatively reflected in Fig.3.1(d), which shows a step decreases of particle velocity near the bottom at the ordering transition. A plausible explanation is that the stronger decay of velocity in the crystalline state is a consequence of the reduction of downward transfer of horizontal momentum due to the formation of hexagonally ordered horizontal sheets (Fig. 4 of Ref. [56]) which slip over adjacent layers in a coherent way. This behavior is consistent with the measured reduction of shear force. Also included in Fig. 5.3 is the velocity profile of a thicker packing (in this case 36 layers), which reaches a stationary state with its degree of local crystalline order gradually varying with height (more ordered on the lower part); in this case, the velocity profile of this non-uniform state lies between the uniformly ordered and disordered states.

The smooth sidewalls can induce crystalline order that is localized to their vicinity and oriented quite differently than the horizontal planes of the globally crystallized state, as is revealed by the internal imaging at different planes. For example, in the images for calculating the disordered-state velocity profiles in Fig. 5.3, this hexagonal structure is visible at one vertical plane that is closer to the sidewall than the other plane, in which grain configuration appears to be random. In Fig. 5.3, we choose the bin width for the disordered state in such a way that each of the triangles corresponds to one row in the 22-layer side-wall induced structure. Nevertheless, the presence of this sidewall induced local order does not have a strong impact on the velocity profile, because the sidewall induced structure is not oriented in a way that favors slip between layers oriented perpendicularly.
to the imposed velocity gradient. This is a useful example showing that the local rheology of granular flows is highly anisotropic.

5.1.2 Effects of system size and particle size

In this subsection, we investigate the development of the non-linear velocity profile with increasing packing thickness, and the effects of particle size. We limit our examples here to uniformly ordered states to maintain comparability. The fact that the crystallized packing flows like stacked semi-rigid horizontal sheets (Sec. 5.1.1) with a vertical profile of velocity insensitive to the choice of measurement plane provides an additional convenience in making comparisons.

We present in Fig. 5.4 the time-averaged velocity profiles measured on an internal vertical plane (xz), using different amounts of glass beads which form crystallized states of 5, 8, 12, 18, and 24 layers, respectively. Each data point represents the mean velocity and the height of a well-defined layer.

Note that in a shear flow as thin as 5 layers of material, the velocity profile is almost linear, approaching zero at the bottom and its maximum value at the top; in this case the shear rate is uniform. However, as the thickness increases, the normalized velocity profile progressively deviates from the linear relation. (This deviation can be seen on Fig. 5.4, by comparing the data points with the corresponding hypothetical linear relation $V(z) = 1 - |z/H_0|$.)

The velocity profiles of flows in these crystallized states gradually approach a master curve that is insensitive to the exact location of the bottom boundary, as thickness is increased. Here the particle diameter $d$ is used to non-dimensionalize the vertical coordinate. The master curve is approximately quadratic on a semi-log plot, as shown on the graph; the fitted velocity profile is equivalent to a Gaussian curve centered at $z \approx +2.2d$. The dominant local length scale of the velocity decay with height can be extracted by defining a spatial decay rate $\lambda(z) = (dV(z)/dz)/V(z)$ and substituting the Gaussian fit for $V(z)$; the result gives $\lambda(z) = (0.103 - 0.02375(2z/d)) \cdot d^{-1}$. (For instance, $\lambda^{-1} \approx 5.1d$ at $z = -2d$; $\lambda^{-1} \approx 2.9d$ at $z = -5d$ etc.) We have also verified and show in Fig. 5.3 that, as long as a specific state of order is created, the roughness of the bottom (ranging from a flat
surface to an irregular surface with a grain-scale bumpiness) does not significantly affect the internal velocity field except for regions within about $5d$ from the bottom. In the 24-layer case, the velocity field of nearly the entire packing is more sensitive to the state of internal order than to the roughness of the bottom boundary. In general, as long as the packing is sufficiently thick, the bottom boundary affects the steady-state velocity field of a sheared packing only by its influence on the routes of evolution that lead to different states of internal order, as described in Chapter 4.

We also measure the time-averaged velocity fields using different sizes of mono-disperse spherical glass beads, as shown in Fig. 5.5. These experiments are performed with a flat bottom boundary: the data points here include flows of three different particle sizes at a comparable layer number (14 or 15), plus a 24-layer uniformly crystallized flow. (The 15-layer flow for $d = 0.68\text{mm}$ is only partially disordered on a flat bottom, as is described in Section 3.3.) Notably, the velocity decay of different particles follows roughly the same trend in terms of physical distance from the driving surface. Even though particle diameter $d$ may seem a convenient length to non-dimensionalize coordinates, the inset shows a clear dispersion among velocity profiles for the same state of order but of different particle sizes, when the coordinate is scaled as $z/d$. For both crystallized or disordered states, and different particle sizes, the vertical decay of particle velocity is generally steeper than that of ordinary fluids in the same channel; the curves for the hypothetical fluid flows have a primary decay length $\lambda^{-1} = (W_0/\pi)$ set by the channel width $W_0$, as derived in Appendix A.2. The implications of these velocity decays are discussed further in Section 5.3.

An additional feature of the velocity profiles for these experiments using a flat bottom is the common trend shared by all data: a slightly up-rising tail occurs near the flat bottom; see Fig. 5.5. The extreme case is the velocity profile for $d = 2.0\text{mm}$, which become flat for the lowest 6 layers. In addition, the comparison of the three velocity profiles of 24-layer crystallized states using different bottom conditions (shown as Fig. 5.3) suggests that the rising tail is unique to the flat bottom condition. The underlying mechanism for this behavior is unclear. Furthermore, inspecting the image sequences and time-resolved trajectories for the 2mm-particle data reveals that the lowest six layers translate as a rigid block. Particles right above the ‘interface’ translate in a stick-slip fashion relative to the rigid block below; the typical steps of the stick-slips are rather large (about $10^4d$).
Interestingly, despite of the dramatic stick-slips between the top of this rigid block and the region above, the vertical profile of time-averaged velocity does not show a discontinuity, except for an abrupt change of the slope on this semi-log plot.

5.1.3 Invariance for changes in driving speeds and normal loads

In Fig. 5.6, we show that the asymptotic mean velocity of grains scales almost perfectly with driving speed over a range of two decades of driving speeds in our experiments, for flows of the same thickness and state of ordering. This behavior is consistent with the notion that the packing is driven in a quasi-static state, for which the coarse-grained local displacement at a given depth is simply proportional to the accumulated displacement of the driving boundary. In Section 5.2.1, we also show that in the speed range of our experiments, the temporal duration of the transient response following reversal of the driving speed also has a simple (inverse) scaling with driving speed.

Our smallest normal load of 1.3Kg\(\cdot g\) is much larger than the total weight of the grains in all cases. In the case of a 24-layer packing, this normal load is more than ten times the net weight of the grains immersed in the fluid. We also vary the normal load by about a factor of 4 and measure the velocity profiles, shown as Fig. 5.7; the invariance of the velocity profiles suggests that the normal load is sufficiently large that gravity plays an insignificant role in its dynamics. However, our largest normal load is much smaller than the load required to significantly deform or damage the glass beads, which could result in stochastic internal shear bands as reported in simulations [1] and experiments modeling geological systems. The change of overall thickness caused by varying the normal load is less than 1 percent.

5.1.4 Effective macroscopic friction

Within the range of our experimental parameters, the boundary shear force is almost proportional to the applied normal force, and does not show detectable change with respect to change of driving speeds over two decades. The effective macroscopic friction coefficient, defined as the ratio of shear stress to normal stress, is about 0.31 for the disordered
state and 0.26 for the crystallized state—see Fig. 3.1 for the change at the crystallization transition. These values are close to the typical values reported in the simulation of Ref. [1], but significantly smaller than the friction coefficients shown simulations reported in Ref. [22] and Ref.[59]. The discrepancy may be partly due to different settings of material parameters. Also, by comparing the measured shear force between dry-particle and immersed-particle experiments, we find that the presence of the immersion fluid reduces the macroscopic shear force by about 12 percent in our experiments, presumably due to lubrication.

5.2 Transient response to shear reversals

To obtain deeper insight into the quasi-static grain dynamics, we extend our investigation from steady states to transient responses. Here we focus on a particular type of transient—the brief period during which a sheared packing readjusts its internal structure to a new steady state after the direction of shearing has been reversed. During this period, grains appear anomalously mobile compared to their steady-state behavior. We refer to this behavior as a shear-reversal transient, which may be regarded as a brief period that the material is “soft”. The investigation into the transient behaviors of dense granular flows may provide useful information for going beyond a mean-field description.

5.2.1 Grain motions during the shear-reversal transient of a thick packing

For either the disordered or the crystallized state, one can stop the steady shearing and then resume the boundary motion in the opposite direction. The reversal of shear causes extraordinary motions of internal grains, and successive sinking of the driving boundary under a fixed normal load. In this section, we use a typical packing thickness (200g of glass beads of diameter $d = 0.68$mm, which would form a 24-layer structure when crystallized) as an example to illustrate the generic behavior of the shear-reversal transient.

Fig. 5.8 demonstrates the displacements of individual grains sampled on an internal vertical (xz) plane of a disordered state, whose boundary motion is being switched from +x to -x. During the first 12$d$ of travel of the boundary motion in the -x direction (b),
the particles move significantly more, compared to the displacements during the same amount of boundary motion in the prior steady state (a). However, unlike the behavior of an avalanche, all grains are still slower than the speed of the driving boundary. The displacements during the next $12d$ of boundary motion in -x direction (c), become closer to the amounts shown in (a). The amplitude of response decays with height, but remains detectable down to the bottom of these images. By inspecting the sequential images frame-by-frame at a time interval corresponding to 0.1$d$ of boundary displacement, we find no detectable propagation delay through the entire image. Furthermore, at any stage of the process including the middle of this transient, when the boundary motion is suddenly stopped, all grains stop essentially simultaneously within the same brief time interval. The lack of detectable propagation time for either the start of the stop of the grain motion suggests that the transient behavior is still a quasi-static behavior, where the role of grain inertia, if any, is negligible. For convenience, we call the behavior during this first $12d$ of boundary translation in the reversed direction a “soft period” of the flow.

Fig. 5.9 shows another measurement performed with a crystallized state. With repetitive reversals of boundary shearing, we compute the average velocity of about 60-80 grains on several horizontal planes as functions of time; the graph shows that the velocity peaks are nearly simultaneous throughout the entire packing and are well synchronized to the reversals of the shearing. The amplitudes of the peaks decrease with height, in a manner that is consistent with Fig. 5.8. The graph also includes an additional test confirming that the velocity peak does not occur if the boundary driving is resumed in the same direction. As the vertical distance from the shearing surface increases, the amplitudes of the transient peaks gradually decrease to the noise level of about $10^{-2}$. (The noise level of $10^{-2}$ in Fig. 5.9 is consistent with a simple estimate obtained from the uncertainty in particle positions and the number of particles being averaged.)

The larger stochastic noise visible at $z = -7.5d$ in Fig. 5.9 is not measurement noise, but rather a manifestation of the collective intermittency illustrated in Fig. 5.2. Furthermore, inspecting the data of similar experiments with higher time resolution also reveals that the observed temporal width of the shear-reversal transient does not depend significantly on the height of measurement.

We also vary the driving speeds by two decades, with the same protocol of shear
reversal as shown in Fig. 5.10. The time step in all three graphs corresponds to the time required for the boundary to translate a distance $0.096d$. Despite of two decades of change in the actual time scale, the peak widths (measured in terms of boundary displacement rather than time) are essentially the same for different driving speeds. Therefore, the rate-independence is true not only for the steady-state velocity field (as shown in Sec. 5.1.3) but also for the time-dependent transient displacement during flow reversal.

In our shear-reversal experiments, the driving protocol features a static period (typically 1s or 10s) during which the boundary is completely at rest; the shear stress is abruptly relaxed to zero near the end of this static period - there is a short period (typically 0.1s or 1s) during which the shear stress is essentially zero. We find that the grains start to move significantly at the moment when the motion of the boundary begins in the opposite direction, rather than at the moment of the release of shear stress. This observation suggests that the major effect reported here is not an energy exchange with a conservative energy storage mechanism such as gravity or the elastic deformation of grains. Instead, the extra mobility during the shear-reversal transient reflects the adjustment of the internal texture in order to adapt to a new direction of shear. The adjustment involves a gradual quasi-static reconfiguration of the contact network.

It is understandable that a sufficient amount of total boundary displacement is required for the packing to fully adapt to a direction of shearing. In Fig. 5.11, we demonstrate that, if the shearing in one direction is not sufficiently long before the next reversal, the next reversal transient can be significantly smaller, reflecting the reduced development of internal texture during the shearing in the previous direction. In general, we find that a total boundary translation of $10^2d$ is needed for the full development of the packing in each shearing direction, in the sense that the grains then respond to the shear reversal with an amplitude and shape that is insensitive to the shearing accumulated previously. (All shear-reversal curves shown in this section are measured under the condition that the boundary has continuously traveled in one direction for at least 400 in each graph.) We can probably regard this $10^2d$ of unidirectional shearing as a process of “anisotropic hardening” towards a specific direction. The “soft period” can be also regarded as the transition from one “hardened” state to the other.

The obvious difference between the curves of internal grain velocity $V_x(t)$ and that of
driving velocity $U(t)$, as shown in Fig. 5.9 and other figures reflects a complicated non-linear relation between the local average displacement and the displacement of the driving boundary. Unlike the case in conventional non-linear visco-elasticity where there is a small amplitude below which a linear description can be a starting approximation of the material response, no linear regime can be defined in describing this important “soft period”, that reflects the reconfiguration of the internal network to accommodate the new direction of shearing. Clearly, to go beyond a mean-field description of the grain motion inside dense granular materials, the development of more advanced theoretical tools that do not require a linear regime as the starting approximation are necessary.

### 5.2.2 Change of volume in response to shear reversals

Repetitive reversals of shearing result in significant sinking of the upper boundary that is driven under a fixed normal load. Fig. 5.12 shows the vertical displacement of the upper boundary measured once during each reversal of its direction of motion. The successive sinking occurs in both crystallized and disordered states; in fact, the total sinking caused by multiple reversals of shearing can be more than the amount caused by the change of bulk density due the structural change in the interior. For either the crystallized or disordered state that has been sufficiently compacted by a long-term shearing (driving in one direction for more than $10^5 d$), applying the repetitive shear reversals (oscillatory shearing) does not change the internal order significantly; the packing tends to gradually recover to the previous stationary volume when the unidirectional shearing is resumed, shown as the data following $t = 14850s$ or $t - t_0 = 1000s$. These facts suggest that the volume compaction induced by shear reversal is highly non-uniform, and primarily localized in the upper layers. Furthermore, when the upper boundary starts to move in a reversed direction, its height makes a sudden descent of about $d/5$, followed by the much more gradual recovery described above. Although the sudden descent is not shown in Fig. 5.12 in which the data are collected discretely (once every revolution of the rotating boundary), this phenomena can be demonstrated by the continuous sampling of the boundary height as the shearing reverses. These time-resolved measurements of the change of boundary height can perhaps be used to test theories or computer simulations of packed grains in a
rather direct way.

5.2.3 Development of shear-reversal transient with packing thickness

We find that the occurrence of the shear-reversal transient, which may be coined as a “soft period”, depends on having a packing that is sufficiently thick. In this section we report the internal response to shear reversal at different packing thickness $H_0$. In Section 5.2.1, we have established that the anomalous mobility occurs throughout the entire packing with negligible propagation delay when the motion of the upper boundary is reversed. In the experiments shown in Fig. 5.13, we use the time-resolved velocity measurement at the mid-height as the indicator of the occurrence of the anomalous mobility. The top row of Fig. 5.13 shows the protocol of boundary driving that is shared by experiments of different packing thickness. The anomalous mobility becomes identifiable as the packing thickness increases to 13 layers or larger. These experiments are performed using the typical particles with $d = 0.68$mm. The characteristic duration of the anomalous motility, when it occurs, is insensitive to the packing thickness. The typical duration is labeled as $\tau$ on the graph and corresponds to about 10$d$ of the translation of the driving boundary. On the other hand, by using progressively larger particles with the vertical dimension kept fixed as 15 layers, we find that the anomalous mobility become less significant and eventually undetectable when the particle diameter $d$ is as large as 2mm. Therefore, sufficient numbers of layers in both vertical and horizontal dimensions ($H_0/d$ and $W_0/d$) seem essential for the phenomena of this “soft period” to occur.

In Fig. 5.4(b), we quantify the contrast between the transient stage and the post-transient stage by time-averaging the velocity; the deviation of the two values from each other marks the characteristic layer thickness above which the shear-reversal transient is observed. The average value of the post-transient velocity serves as an estimate of the true asymptotic steady state velocity at the mid-height $(z = -H_0/2)$ for different packings; at large thicknesses, this value reflects the decay with $z$ that is consistent with the master curve [Fig. 5.4] of the steady-state velocity profile for a sufficiently thick packing. (Compared to Fig. 5.4, the larger error bar here and its non-smooth variation with $z$ is due to insufficient total sampling time– a compromise for obtaining the required time-resolution.) In addition,
the typical local displacement accumulated during the transient, as displayed at the right axis of Fig. 5.13(b), shows a different steepness than the decay of the steady-state mean velocity with observation height (in this case $-H_0/2$).

It would be very interesting to construct a minimal theory to predict the development of this shear-reversal transient as the layer thickness is increased, as well as the duration of the transient (about $10^1d$ of boundary displacement).

5.3 Discussion

5.3.1 Shear banding

In this section, I discuss on the origin of shear banding, i.e. the localization of velocity to a narrow region (the so-called shear band), and the factors that affect the steepness of the decay of particle velocity.

The sheared granular packing studied in our experiments exhibits shear banding at the upper driving boundary, accompanied by a continuously decaying velocity profile with a finite slip at the stationary bottom; see Fig. 5.3, Fig. 5.4, and Fig. 5.5. Though the internal ordering of grains alter the transverse velocity profiles at each horizontal plane, the global trend of shear banding with respect to height is shared by both the ordered and the disordered states of flow. Shear banding is commonly observed in both experiments and simulations of granular flows: the shear band more often occurs in the vicinity of a driving boundary [36, 5] or under a free surface [27]; it can also be found in the deep interior of a granular flow [16], depending on the global geometry of a model system and other parameters such as the stress level [1]. Recently, a theoretical study [58] proposes a way to determine the location of the shear band in the situation described in Ref. [16], by treating it as a variational problem that minimizes the frictional dissipation. In this theory, the shear band is treated as a mathematical surface whose thickness is infinitesimal.

From a purely theoretical point of view, a two-dimensional (2D) packing with no gravity, sheared by two identical parallel boundaries moving steadily relative to each other, has reflection symmetry with respect to the mid-plane between the two boundaries. If a theory would predict a unique steady-state profile for this 2D problem, the predicted profile must have the same reflection symmetry. A linear velocity profile is one possibility (although
not the only one), while a one-sided shear band clearly does not satisfy the reflection symmetry. For instance, the simulated steady-state velocity field of a thick 2D granular shear flow as reported in Ref. [59] would approach a linear profile [57], if the conditions are set to be symmetric by turning off the gravity (which was non-zero and pointed towards one of the shearing boundaries in the situation reported in Ref. [59]). The alternative theoretical scenario for an asymmetrical steady-state profile in a symmetrical 2D problem is to allow non-unique stationary velocity profiles that are selected by prior history (initial condition) of the flow - see the simulated velocity profiles reported in Ref. [1]. Theories of this type would need to contain internal state variable(s) other than the velocity gradient in order to register the effect of prior history on the evolution of the velocity field. Beyond the idealized 2D scenarios described above, the commonly observed asymmetrical shear banding is totally reasonable if the effects of sidewalls are included using a full 3D theory. The author is not aware of such a 3D theory that can be directly applied to our sheared packings in the annular channel.

Our experiments of sheared granular packing do not satisfy the reflection symmetry of a simple 2D problem. In the following discussion, I examine four factors that break the reflection symmetry: (1) the weight of individual grains under the influence of gravity, (2) drag force from the interstitial fluid, (3) the initial state at which all particle are at rest, and (4) the stationary sidewalls attached to the bottom. The grain weight and fluid drag (1,2) are insignificant factors, since it can be estimated that our large normal load creates inter-particle compression forces that are a few orders of magnitude larger than individual grain weight and the maximal viscous drag by the fluid. In Hanes and Inman’s experiments [19] using an annular shear channel comparable to our setup except with their upper boundary being at rest, the shear band always occurs near the upper stationary boundary. This observation suggests that the initial velocity distribution of grains alone is not sufficient for creating a ‘preference’ for nearly zero velocity in the bulk and a shear band located at a mobile boundary. The velocity field measured across the channel at a constant height (Fig. 5.1) shows a significant drag by the sidewalls. The analyses above suggest that the stationary sidewall (4) seems to be the dominant \(^1\) factor in determining

\(^1\)Although the initial velocity distribution of grains may not affect the trend of shear banding, the initial state of packing may influence the evolution of the packing and the selection between different states of
the general trend of the vertical profile (shear banding). In Section 5.3.5, I use a heuristic model to explain that shear banding can emerge as a consequence of resistance exerted by the sidewalls.

5.3.2 Characteristic decay length of velocity field

The steepness of the spatial decay of velocity field in a shear flow is often characterized by an exponential decay length, sometimes called the width of the shear band. The characteristic length is influenced by both the properties of grains and geometrical factors of the system. First of all, we demonstrate in Fig. 5.3 that the state of internal order decisively affects the steepness of the velocity decay. The decay lengths derived from the master curve for sufficiently thick packings (Fig. 5.4) are apparently consistent with the conventional wisdom that the width of a shear band at the vicinity of the driving boundary is comparable to a few particle diameters. However, Fig. 5.5 shows that the velocity profiles of different sizes of particles clearly do not fall on the same curve when the distance from the shearing surface is scaled in units of particle diameters. Rather, the trends of velocity decay for different bead sizes seem similar in terms of actual physical distance. These observations suggest that the spatial decay rate of particle velocity is controlled primarily by the channel geometry, while the exact steepness of the decay can be fine-tuned by the internal structure of the sheared packing.

The experimental evidence that we and other researchers have reported suggest that the spatial decay of velocity as a function of distance from a driving surface in a boundary-driven granular shear flow can be geometry-specific and strongly influenced by the interaction with the stationary walls that confine a granular packing. In fact, there are no obvious theoretical reasons explaining why the decay length should depend explicitly on the grain size. It may not be surprising that the measured decay curves in different experimental granular flows not only show a large dispersion of values in units of grain diameter, but also show differences in functional forms.

For instance, the velocity decay lengths in units of grain diameter obtained from the interior of a thin Coutte-cell flow driven by its inner cylinder [36] are between 2 and 1, lower than the typical values extracted from our velocity fields near the driving boundary. order(explained in Chapter 4); therefore the initial state can still affect the exact velocity field indirectly.
Nasuno and coworkers [27] determined the spatial decay of the creeping velocity far below an inclined surface of flowing grains, by monitoring the grain motion visible through the transparent sidewall: the vertical profile they obtained shows a functional form that is almost purely exponential, quite distinct from the velocity profiles reported in Ref. [36] and in our work. But note that the grain motion measured in Ref. [27] are under the direct influence of the sidewall, as opposed to the internal measurements in our work and Ref. [36]. This factor may be responsible for distinct functional forms of velocity decay.

Would a shear flow experiment approach a quasi-2D problem as the cross-sectional aspect ratio goes to extreme? In our system, when the packing thickness $H_0$ is small compared to the fixed channel width $W_0$, i.e. $(H_0/W_0) \rightarrow 0^+$, the flow can be considered as a quasi-2D limit, but is perhaps a trivial one. The velocity field of a 5-layer in Fig. 5.4 exhibits a nearly linear vertical profile. Note that this linear flow limit does not reflect anything about the sheared material - an ordinary fluid in a the same channel with a small $H_0/W_0$ would behave the same way. On the other hand, the system is not designed in a way that $W_0$ can be increased independently to test the limit. On the other hand, the standard Coutte-cell experiment with a rotating inner cylinder (in the vertical direction) is commonly believed to be close to a quasi-2D problem when the total filling height of the material ($L_F$) is large compared to the gap width ($\Delta R$). But note that the asymmetry caused by the open upper surface, the vertical gradient of compressive stress inside the material, and the bottom condition can have significant effects on the flow. (Ref. [16] illustrates the long-range influence of the conditions on the bottom.) It is relatively inconvenient to vary $\Delta R$ continuously to study the possible influence of packing thickness on the velocity gradient, compared to our system (as shown in Fig. 5.4). More importantly, the commonly observed fact that granular Coutte flows with an extremely small aspect ratio ($\Delta R/L_F$) still exhibit shear banding as opposed a symmetric linear profile can involve an intrinsic asymmetry in this geometry: the internal shear stress ($\sigma_{\phi r}$) varies as $\sigma_{\phi r} \propto 1/r^2$ as a natural consequence of the conservation of angular momentum. This variation of stress may play an important role in the spatial decay of velocity, particularly for quasi-static granular flows, as we discuss further in Section 5.3.5.

To test theoretical ideas that may integrate the experimental information of dense granular shear flows up to date, the analyses in this subsection indicate that implementing
a theory in a 3D context is necessary. A decisive test of theories would be to see whether incorporating the interaction of the sidewall in various experimental geometries can produce the observed general trend of the velocity field (shear banding).  

In the preceding discussions, the curvature of the circular channel in our system and other Coutte-cell experiments are generally ignored. Circular geometries are advantageous for eliminating “end effects” and are commonly used in both 2D and 3D experiments. But in view of the long-range “force chains” (as can be visualized by the use of photo-elastic materials as in Ref. [24]), the finite radius of curvature may still play a minor role in the flow, especially for quasi-statically packed grains.

5.3.3 Development of bulk properties with packing thickness

Several notable bulk properties observed in our system occur only if the packing thickness is sufficient:

Distinct stats of order – In the experiments of our typical particles with \( d = 0.68 \text{mm} \), we find that about 15 layers of materials or more are needed for developing two clearly distinguishable states of order. The states of internal order for experiments with less amount of glass beads depend on the exact condition of the bottom surface: the degree of order can be highly sensitive to layer thickness (Fig.3.4), or can be inhomogenous around the channel (with the shear force close to the typical value for a disordered state. For packings thicker than 30 layers, spatially non-uniform states with a detectable vertical gradient of local order can be created and persist indefinitely. (Please see Section 3.2, Section 3.3, and Fig. 5.3 for these observation.) Therefore, the “available phase space” of the system in terms of its state of order depends on system size; our typical packing size (24 layers with \( W_0/d = 28.5 \)) lies within the range that a simple two-state scenario is sufficient.

\[2\text{In units of particle diameter, the spatial decay of velocity field produced in existing 2D theories and simulations such as Ref. [59, 1] generally have a length scale an order of magnitude larger than all experimental measurements reviewed here.}\]
Anomalous mobility in response to reversals of boundary motion – The “soft period” upon the reversal of shearing, i.e. the anomalous mobility during the shear-reversal transients requires at least 13 layers in thickness for the phenomena to develop fully (Section 5.2.3).

Deviation from a linear velocity profile (Shear banding) – We demonstrate in Fig. 5.4 that the vertical velocity profile progressively deviates from a linear profile (for 5-layer flow) to a non-linear master curve (for 12 layers or more). This behavior may involve not only the sufficient development of bulk property, but also the crossover of relative importance of the sidewalls and the rough bottom.

5.3.4 Rate-independent frictional dynamics

The grains in the entire sheared packing are in a quasi-static regime, as pointed out in Appendix A.1. In this regime where particles are in contact with multiple neighbors, friction is expected to be an important parameters of the problem. Comparing behaviors of dry particles and fluid-immersed particles (Fig. 3.3) suggests that changing the microscopic friction substantially changes the internal velocity fields of particles: without the lubrication introduced by the immersion fluid, the precursor before the crystallization transition is found to be an order of magnitude longer (under the same driving speed and boundary conditions). On the other hand, we find that the lubrication introduced by the fluid reduces the shear stress at the driving boundary only by about 12 percent (Sec. 5.1.4), under a fixed normal load. This observation implies that the shear stress detected at the driving boundary is transmitted mainly through the particle-scale surface irregularities, rather than microscopic friction forces that are tangent to the contact area.

Over the entire two decades of driving speeds, we do not find properties that are dependent on shear rate; this is consistent with the notion that the sheared grains are completely under the quasi-static regime. We do not find rate-dependent phase transition or ‘shear-melting’ as in some other systems, which often reflect the competition between the rate (or energy) of imposed shearing and the rate (or energy) of local activity associated with bulk excitation (such as mechanical vibration or thermal motion). At the level of compression stress imposed by the normal load (minimally (1.3Kg)/g), the entire range of
applied shear rate is too small (such that \( X << 1 \) in Appendix A.1) compared to the minimum that is required for any individual grain to be temporarily independent of its neighbors.

As a result, the underlying physics of the dynamics of sheared quasi-static packing studied here is perhaps mainly about the adjustment of a packed structure driven under boundary constraints, as opposed to reflecting instantaneous competitions between different factors. The quasi-static adjustments are rate-independent and often irreversible. In Section 5.2.1, we illustrate the rate-independent adjustment of internal grains to a new direction of flow, in the form of an anomalous mobility during the shear-reversal transient. A fully non-linear model (that does not require a finite linear regime) may be necessary for describing this behavior. Interestingly, densely packed grains may adjust to alternating shearing with a fixed amplitude, as well as to a stationary shearing in a fixed direction - see the experiments reported in Ref. [42] as an example, where grains show stepwise descents in heights only when the amplitude of imposed cyclic shearing is abruptly change.

### 5.3.5 Granular flows vs. Fluid flows

How does the velocity field of a slowly sheared granular packing compare to a viscous fluid driven in the same geometry?

In Appendix A.2 we discuss the consequence of the fact that, in the simple case of an incompressible Newtonian fluid, the velocity field for the steady flow satisfies a 2D Laplace equation \( \nabla^2 v_x = 0 \) across a rectangular channel. This equation couples the second derivatives (the “curvatures”) in two orthogonal directions. The vertical decay lengths are well coupled to the horizontal wave numbers that are selected by the finite-width channel due to the no-slip boundary conditions. The principal decay length is therefore \( 1/\lambda_1 = (W_0/n\pi)|_{n=1} = W_0/\pi \). (See the slope on the semi-log plot Fig. 5.5). Note that this is exact only when the fluid is Newtonian. Otherwise, the more general stress balance (as in Eqn. A.7) requires

\[
(\nabla \eta) \cdot (\nabla v_x) + \eta \nabla^2 v_x = 0 \tag{5.1}
\]

on the \( yz \) plane rather than \( \nabla^2 v_x = 0 \). Here, one can see that the two “curvatures” of the velocity field are comparable \( \left( \frac{\partial^2 v_x}{\partial z^2} \approx -\frac{\partial^2 v_x}{\partial y^2} \right) \) when the non-Newtonian term of the
equation can be ignored - this is true only when the characteristic length scale of variation in $\eta$ is much longer than the cross-sectional dimension of the channel. Therefore the characteristic vertical decay length may still be comparable to the channel width if the fluid is only weakly non-Newtonian.

On the other hand, granular particles slip against the smooth sidewalls instead of satisfying no-slip boundary conditions for ordinary fluids. This observation may suggest that granular flow would not be slowed down by the sidewalls as much as ordinary fluids driven in the same channel. However, by comparing measured velocity fields of granular particles to the curves representing the ordinary-fluid flows (Fig. 5.5), we find that the downward decay of average velocity at each height for granular particles is indeed much steeper than that of ordinary fluid. This apparent conflict is resolved if one recognizes that granular flow is highly non-Newtonian so that the vertical ($z$) and horizontal ($y$) variations of velocity field do not need to be comparable. A quasi-statically sheared granular packing is obviously non-Newtonian, since the shear stress only has a very weak dependence on shear rate, if not vanishing. This fact is confirmed by our shear force measurements with different driving speeds, and by numerous previous experiments.

In the following discussions, I attempt a heuristic model showing that, if the internal shear stress of a material has only a weak dependence on the local shear rate, shear banding can occur under the influence of the frictional resistance imposed by the sidewalls. The simple model is based on coarse-grained assumptions of local rheology, and does not explicitly involve the size of particles.

Here we assume that the shear stress inside the material is related to the local shear rate. And we restrict the discussion to the cases where the shear stress monotonically increases with shear rate; this restriction ensures the stability of a stationary velocity profile inside the bulk.

We use the crystallized state as an example, for its simple feature that the velocity is uniform at each height. The shear stress $\sigma_{xz}(z)$ and the steady-state velocity $V_x(z)$ are therefore functions of height $z$ only. The shear rate $\dot{\gamma} = V'_x(z)$ is simply the derivative of $V_x(z)$. The $z$ axis points vertically upwards with $z = 0$ representing the driving boundary at the top. The shear resistance exerted by the sidewall per unit area is represented by
\( \sigma^{Wall}(z) \). By considering the stress balance for a stationary state, we have

\[
W_0 \cdot (\sigma_{xz}(z) - \sigma_0) = 2 \int_0^z \sigma^{Wall}(z')dz'
\]  

(5.2)

in which \( \sigma_0 \equiv \sigma_{xz}(0) \). In [1] and [2] of the following discussions, we ignore the finite height \( H_0 \) by assuming \( H_0 \to \infty \) and accepting the fact that \( V_x \) decreases as \( z \) goes away from the driving surface \( z = 0 \).

[1] To the first approximation, we assume that the resistance at the sidewalls is a constant

\[
\sigma^{Wall}(z) = \sigma^W_0
\]  

(5.3)

so that the right-hand side of Eqn. (5.2) reduces to \( 2\sigma^W_0 z \). In general, when one assumes a local relation \( \sigma_{xz} = f(\dot{\gamma}) \) where the value of \( f(\dot{\gamma}) \) slowly increases with increasing \( \dot{\gamma} \), the solution of Eqn (5.2) leads to a shear-banding velocity profile with its gradient highly localized near \( z = 0 \).

For instance, \( f(\dot{\gamma}) \) can be a logarithmic function of shear rate \( \dot{\gamma} \)

\[
f(\dot{\gamma}) \approx \sigma_0 (1 + \alpha \ln(\dot{\gamma}/\dot{\gamma}_0))
\]  

(5.4)

in which \( \gamma_0 \equiv V'_x(0) \). The positive constant \( \alpha \) is assumed to be much smaller than unity so that \( 1 + \alpha \ln(\dot{\gamma}/\dot{\gamma}_0) \) remain positive for the entire interested range of shear rate. (The notion of logarithmic dependence has been put forth by previous researchers; see Ref. [20].) Under these assumptions, one would obtain

\[
V_x(z) = V_x(0) \cdot \exp \left( \frac{2\sigma^W_0}{\alpha \sigma_0 W_0} z \right)
\]  

(5.5)

which exhibits an exponential shear banding. Note that the decay length of this velocity profile is proportional to \( W_0 \), but is scaled down by a factor of \( \alpha \).

[2] An improvement of the model is to introduce a spatial variation of the resistance force at the sidewalls. In general, if \( \sigma^{Wall}(z) \) gradually increases as \( z \) goes downwards, we can expect an increasingly negative slope on the semi-log plot of velocity profiles such as Fig. 5.4 and other figures. One possibility is to assume a “velocity weakening” frictional law \( \sigma^{Wall} = g(v^{slip}) \) with a logarithmic dependence on the slip velocity \( v^{slip} \)

\[
g(v^{slip}) \approx \sigma^W_0 (1 + \beta \ln \left( v^{slip}/v^{slip}_0 \right))
\]  

(5.6)
in which a small negative $\beta$ characterize the slight decrease of force with respect to the slip velocity. For our crystallized state, we can assume $u_{slip}(z) = V_x(z)$ and make the problem self-content. Since there is no analytical solution to the resultant differential-integral equation [Eqn (A.9) in Appendix A.3], an approximate solution

$$\ln(V_x(z)/V_x(0)) \sim \frac{2\sigma_0^W}{\alpha\sigma_0 W_0} \cdot (z + \frac{\beta}{2} \frac{2\sigma_0^W}{\alpha\sigma_0 W_0}z^2)$$

is derived using the iteration scheme described in Appendix A.3. Note that the quadratic form of this approximate solution qualitatively captures the shape of the master curve shown in Fig. 5.4.  

Finally, to take into account effect of the rough bottom boundary at the finite depth $z = -H_0$, one would need to introduce an appropriate boundary condition that “penalizes” slip against a rough surface, i.e. the resistance increases with larger slip velocity. I expect that an additional ‘velocity-strengthening’ boundary condition for a rough bottom should lead to a steeper velocity decay as the value of $H_0$ decreases and, in the extreme case, a linear profile when $H_0/W_0$ is sufficiently small (as shown in Fig. 5.4). In principle, a condition that suppresses slip at the rough boundaries at the upper and the lower surface should also be important for justifying the stability of the profile. We may speculate whether the observed stick-slips in the experiment using a flat bottom (described in Section 5.1.2) can indeed be the consequence of not having a rough bottom that penalizes stick-slips and stabilizes the velocity profile near the bottom.

More generally, if the constraint of a monotonic increase for the function $f(\dot{\gamma})$ is lifted, a smooth velocity profile can be unstable and abrupt internal shear bands may occur (such as the case in some 2D particle dynamics simulations [1]). The measured velocity field in our system are all continuous, without showing discontinuities associated with internal shear bands. Whether this experimental fact sets a constraint on the possible rheological models for quasi-static grains, or is due to the difference between 3D flows and 2D flows, needs further clarification. Finally, we remark here that the simplistic theoretical pictures

\[A\] direct comparison between the simple approximation Eqn (5.7) and the quadratic fit for the master curve shown in Fig. 5.4 give $\alpha = (0.103)^{-1} \cdot 2(\sigma_0^W/\sigma_0)(0.68mm/19.4mm)$, and $\beta = -2(0.0237)(0.103)^{-2} \approx -4$. The result for $\alpha$ is reasonable as long as $\sigma_0^W$ is comparable to or smaller than $\sigma_0$, but the value for $\beta$ is unsatisfactory. Further improvements of the approximation scheme, or of the simple assumptions made the internal rheology and sliding friction at sidewalls [Eqn (5.4) and Eqn (5.6)], may be in need.

\[B\] One possibility for establishing a boundary condition that suppress slip is to adapt Eqn (5.6) except for using a positive $\beta$ at a rough surface.
discussed in this paragraph have not yet included any state variables that register the history of the packing, which we find essential for accounting the evolution of crystalline order.

It may also be interesting to see if one can find the similarity between the flow field of the sheared granular packing and that of a strongly shear-thinning fluid\(^5\) in the same channel, since the latter may approach the case where shear stress has only a very weak dependence on the local shear rate. Furthermore, a disordered sheared packing may be compared to an isotropic fluid, while the crystallized flow clearly requires a model that exhibits anisotropy in its local rheology.

---

\(^5\)Shear thinning is usually characterized by the behavior that, at high shear rate $\dot{\gamma}$, the shear stress $\sigma \propto \dot{\gamma}^n$ with $n < 1$; therefore the apparent viscosity $\eta \equiv \sigma/\dot{\gamma}$ decreases as the shear rate increases. (This covers a wide range of materials that can either become Newtonian ($n = 1$) at low shear rates or has a yield stress.) While an analogy to a granular flow may require $n \to 0^+$ (which is often coined as “plastic” material in the sense that its shear stress is insensitive to shear rate once it yields), real fluids that have an $n$ below 0.4 are considered rare in practical circumstances.
Figure 5.1: The three-dimensional structure of the velocity fields. (a) Velocity profiles before crystallization; (b) Velocity profiles after crystallization. The packing is driven at a speed $12d/s$ with $d=0.68\text{mm}$. Starting from a disordered state, the precursor prior to crystallization extends a few hours. The $x$ axis refers to the direction of the mean flow. The $z$ axis sets the height coordinate with the upper boundary defined as $z = 0$. The coordinate $y$ indicates the distance from the outer wall of the channel. The graphs on the left show the horizontal profiles measured at the mid-height of the sheared packing before and after the crystallization, with a sample horizontal slice image shown as inset in either case. Different data symbols represent velocity fields measured at different times, about half an hour apart. The graphs on the right show the vertical profiles measured at different distances ($y$) from the outer wall, for about the upper one-third of the packing. Measurements at different $y$ are performed within a period that is negligible compared to the typical time of variation in velocity fields, and are therefore essentially simultaneous. A constant-velocity contour on a cross-section is sketched schematically, for each state of order. Images and velocity profiles show that particles of the crystallized state moves almost as a rigid sheet while the velocity field of the disordered state shows a significant decay towards the stationary walls, and that the crystallized state has reached a stationary velocity profile as opposed to the disordered state that is still evolving.
Figure 5.2: Time-resolved particle trajectories sampled at horizontal (xy) internal planes, with a sampling window of size $8d$ by $8d$ and a depth of field of about $1d$. There are approximately 30-50 particles captured at each instant and displayed using different colors. (a) Transverse positions of particles in a disordered state; (b) transverse positions of particles in a crystallized state; and (c) longitudinal positions of particles in a crystallized state. The scale for the axes for particle coordinates are identical for the three graphs, with the particle size $d$ indicated on the lower left. Strongly collective motion is obvious in the crystallized states (b) and (c) but not in the disordered state (a). (The scale bars in (b) and (c) mark the characteristic period of the collective motion.)
Figure 5.3: Vertical profiles of steady-state velocity for different states of order and with different boundary conditions. Velocities are normalized by the speed of the upper boundary near $z = 0$. For the 24-layer crystallized states, the data marks also indicate the mean position of particles for each well-defined layer; these data points are also fitted by a quadratic curve shown in gray. The scale $\delta$ on the lower right represents the scale of stochastic variation explained in the text. For a crystallized state, the decay of velocity with height is significantly steeper than that of the disordered state. (The triangles and squares represent disordered-state velocity profiles measured during the same experiment but at two different vertical planes that are, respectively, 1/4 and 1/3 of the channel width from the sidewall.)
Figure 5.4: Vertical profiles of the steady-state velocity, measured from packings of different thickness $H_0$, demonstrating the approach to a single master curve for deep layers. Time-averaged velocities $<V_x>$ are normalized by the speed of the driving boundary and plotted against vertical position $z$, measured in units of particle diameter $d$. The interface between the driving boundary and the grains is defined as $z = 0$ in all cases. Hypothetical linear relations corresponding to different thicknesses $H_0$ with the assumption of no slip at the top and bottom, are plotted as reference curves. ($H_0^{(24\text{Layers})} \approx 15\text{mm}$.) The estimated stochastic uncertainty of 30% is shown at the lower right. The measured velocity profiles, with data labels indicating the positions of individual layers, deviate substantially from the plotted linear curves except for the 5-layer flow. At larger thicknesses, the curves approach a master curve that can be fitted by a quadratic curve on this semi-log plot. (Conditions: normal load = (1.3Kg) $g$; driving speed = 12$d$/s.)
Figure 5.5: Vertical profiles of steady-state velocity for particles of different sizes. The time-averaged velocities $< V_x >$ are normalized by the speed of the driving boundary and plotted against vertical position $z$. All experiments are performed with a flat bottom boundary. Horizontal positions of data labels indicate the heights of individual layers, with the exception of the data represented by crosses (which is partially disordered). The quadratic fit curve derived from the data from the velocity field of 24-layer crystallized flow (on a mono-layer bottom) is also shown as light gray. The height $z = 0$ represents the height of the driving boundary in all cases. Theoretical velocity fields of ordinary fluids driven in the same channel, with three different assumed positions of the lower boundary ($z = -10\text{mm}, -15\text{mm}$ and $-20\text{mm}$), are averaged over the channel width and plotted as functions of $z$ as a comparison (described in text). The inset shows the data of the three crystallized states with different particle sizes, plotted against the non-dimensionalized coordinate $z/d$; the three data set shows a clear dispersion when plotted using the non-dimensionalized coordinate.
Figure 5.6: Vertical profiles of normalized steady-state mean velocity, driven at different driving speeds. (Conditions: normal load = (1.3Kg) g ; 24-layer crystallized state, with points marking every layer.)
Figure 5.7: Vertical profiles of normalized steady-state mean velocity for deep layers, under different normal loads. (Common conditions: driving speed $12d/s$; 18-layer crystallized state, with points marking every layer.) The dotted curve represents the master curve derived from profiles of thicker crystallized flows.
Figure 5.8: The trajectories of centers of individual grains during $12d$ of boundary displacement, at different stages of a shear-reversal experiment: (a) steady shearing in the positive $x$ direction; (b) the first $12d$ of boundary motion in the opposite direction (negative $x$); (c) the next $12d$ of boundary motion in the negative $x$ direction following (b). Note the enhanced displacements just after reversal. The driving boundary is located slightly above the top of each graph; the scale of $1d$ is indicated on the left. [Experimental conditions: driving speed = $12d/s$; time resolution = 0.008s. The thickness of the illuminating light sheet gives an effective depth of field (in $y$) of about $1d$; therefore some of the particles that are indeed at different depths may appear closer to each other than $1d$, since this is a two-dimensional projection of particle trajectories.]
Figure 5.9: Time-resolved particle velocity sampled at different heights \((z)\) of the flow, as the boundary velocity \(U(t)\) is repetitively reversed in steps. An extra spike is found at the instant of reversal. The mean velocity \(<V_x>\) is computed by averaging the velocity of all particles in a region \(8d\) by \(8d\) at the specific height, with a time resolution \(\Delta t = 0.2s\). The positions of the sampling plane are indicated in each case, with \(z = 0\) referring to the driving boundary. The response at \(t = 190s\) shows that the velocity peak does not occur if the driving is resumed in the prior direction of shearing.
Figure 5.10: Time-resolved average grain velocity at the mid-height of a 24-layer packing, subject to a fixed protocol of shear reversal but with the driving speed varying by two orders of magnitude. The time step is accordingly adjusted so that $\Delta t$ corresponds to the time for $0.096d$ of boundary translation in all three cases. (The curves have been smoothed by $4\Delta t$ to suppress the measurement noise due to random pixelation.) The three cases exhibit a similar duration of transient corresponding to a few grain diameters of boundary translation, despite huge difference in the actual time scale.
Figure 5.11: The response to shear reversal inside a 24-layer packing. The boundary velocity is shown in (a). The first peak (time 1.5s) is the response after more than $10^2 d$ of accumulated boundary translation in one direction. The amplitude of the second response (time 3.5s) is significantly less than the first peak and those in the previous two figures; this demonstrates that a total boundary translation of about $12d$ is insufficient for the packing to fully adjust to the new shearing direction.
Figure 5.12: The horizontal velocity \( U(t) \) and vertical displacement \( h(t) \) of the upper boundary as the shearing is repetitively reversed. The granular packing has been pretreated by prolonged shearing in a fixed direction. During the repetitive reversal of shearing, the height \( h \) is sampled at the mid-point of the boundary displacement in each direction. Note the successive sinking of the boundary at each reversal and the subsequent recovery to the prior height. (The comparison between the behavior of a disordered and crystallized packing is performed with an identical boundary condition that allows both states to exist, and the same reference is used for defining \( h \); therefore the difference of the asymptotic values of \( h \) reflects the difference of bulk density.)
Figure 5.13: Development of the shear-reversal transients with the packing thickness $H_0$. (a) Time-resolved particle velocity sampled at the mid-height $z = -H_0/2$ of packings of different thickness $H_0$, as the boundary velocity $U(t)$ is reversed after a prolonged unidirectional shearing. The protocol of boundary motion is shown at the top of the graph. The layer thickness is expressed in units of particle diameter $d$ (with the equivalent number of layers indicated in parentheses.) When the packing is sufficiently thick (e.g. 13 layers or more), the shear-reversal transient become identifiable, with a temporal width $\tau$ that is roughly invariant with layer thickness. (b) The average grain speed during the transient period and the longer time interval following the transient period, computed from time-resolved velocity measurements partly shown in (a). The long-time average of grain speeds after the transient can be regarded as an estimate of the asymptotic steady-state velocity at the mid-height $z = -H_0/2$ for each packing thickness. The difference between the average velocity during $\tau$ and the asymptotic velocity can be regarded as an indicator of the excess mobility during the shear-reversal transient.
Chapter 6

Conclusions
In this dissertation, I explore the evolution of internal structure and rheology of a quasi-statically sheared granular packing, and investigate systematically the dynamics of internal grains in the shear flow using variable sample thickness and other parameters. To this end, we built a specially designed annular shear cell, in which the amount of granular material is conserved while its total volume can vary in response to the horizontal shearing under a constant vertical load. Our measurements involve a novel technique of index-matched internal imaging, particle tracking with a wide range of sampling times, and simultaneous detection of granular volume and boundary shear force. This experimental information, as well as the in-depth analyses integrating our results with prior studies, should be useful both for developing theories for flows of quasi-static granular particles, and as guidelines for further experimental studies.

For a thick packing of nearly mono-disperse granular material under long-term unidirectional shearing, the final outcomes can be categorized into two well-defined states: (1) A crystallized state in the form of creeping hexagonal lattice planes oriented perpendicularly to the imposed velocity gradient; the structure is global and persistent, but contains some defects. (2) A disordered state in which no significant spatial order is found in the bulk, except in a few layers near the smooth vertical confining walls. When the system is driven under a fixed normal load, the granular packing undergoes a decrease of total volume with time before settling into either of the two final states. Once sufficiently compacted, both states are stable against perturbations produced by oscillatory shearing.

The different states of internal order result in significant differences in the flow rheology. The crystallization transition causes a significant decrease of both the granular volume and the boundary shear force. Therefore, these two external measurements can serve as alternative indicators of the state of the interior when direct internal imaging is not available (e.g. in systems of dry grains.) Furthermore, by precision measurements of the vertical decay of the asymptotic internal particle velocity, we demonstrate that the vertical decay of grain velocity in the crystallized state is significantly steeper than that of the disordered state. The larger gradient of velocity and smaller resisting shear force in the crystallized state (compared to the disordered state at the same normal load and driving speed) can be consistently explained: the hexagonally ordered layers can slip over each other coherently; this results in a lower resistive force and a stronger reduction of mean
velocity with distance from the driving boundary. This is also demonstrated explicitly by the time-resolved trajectories of multiple particles in each state, and the qualitatively different horizontal profiles of mean velocity fields at a fixed height.

The scenario of the evolution and stability of the two states of internal order under shear driving depends on both the boundary condition of the stationary substrate and the state of the initial packing. If grains are randomly poured into the channel (or fully stirred) with a bumpy bottom, this disordered initial state can be unstable in the sense that a few cycles of oscillatory driving can promptly generate significant partial ordering in the bulk. Driven by a unidirectional shearing subsequently, this partial ordering can either be eliminated or can evolve into a global crystallized state in an apparently stochastic fashion; we surmise that the outcome depends on details of the partially ordered state. On the other hand, once the disordered state has been sufficiently sheared, e.g. by a sufficiently long-term unidirectional shearing with a fixed normal load, the disordered state becomes highly stable—the oscillatory shearing becomes ineffective in generating the partial ordering that can otherwise be induced. This example shows that past history can change the response of the packing to an imposed shearing.

The packing can exhibit a stepwise crystallization transition during long-term shearing, depending on the boundary condition at the bottom of the channel. When either a flat substrate or mono-layer bottom with a quasi-hexagonal structure (which appears to be the most favorable boundary condition for crystallization) is used, the disordered state is generally unstable against boundary shearing and always evolves into a globally crystallized state. When a unidirectional boundary shearing is applied to a thick packing with these substrate conditions, the crystallization transition occurs with a long precursor during which internal grains remain largely disordered; compared to the entire course of evolution, the crystallization transition appears as a relatively sharp change where granular volume, the shear resistance, and the grain velocity profile simultaneously change, along with the degree of internal ordering. The duration of the precursor appears stochastic within an order of magnitude, and is comparable to the characteristic time for the grains near the stationary bottom to move a few particle diameters relative to the boundary, or to each other; this characteristic time varies drastically with the thickness of the packing and scales inversely with the driving speed. The sensitivity to substrate conditions and the
long precursor seems to imply a mechanism similar to epitaxial growth [23, 37]. However, there is no clear evidence that the ordered domain grows from the substrate upwards, because the observed crystallization process is highly non-uniform.

For a sufficiently thick packing with an alternating direction of boundary shearing, our time-resolved particle tracking reveals a brief period (about $10d$ of boundary translation), during which grains exhibit anomalous mobility (compared to the steady-state flow), immediately following the reversal of the boundary motion. This behavior vanishes if the boundary motion is reversed without first persisting in one direction for a sufficient distance, while the amplitude of this transient response saturates once the persistent one-way shearing is sufficiently long. The grain motions during this shear-reversal transient, which is essentially rate-independent and instantaneous throughout the entire packing, do not reflect any intrinsic time scale but only the accumulated displacement of the boundary. The behavior presumably reflects the adjustment of the packing in re-establishing an anisotropic internal contact network, which may reach a saturated state after being sufficiently sheared in one direction. Developing a non-linear model to describe this behavior seems an interesting theoretical challenge.

Several notable bulk properties of the packing are present only if its thickness is sufficient. The observation of a minimum sample thickness may be important in designing model systems of granular flows. Using the typical conditions particle diameter $d=0.68\text{mm}$ (such that channel width $W_0 \approx 28.5d$) as examples: (1) The well-distinguished two states of ordering requires sufficient thickness of the packing. The typical packing thickness used to characterize the two states of order described in Chapter 3 and Chapter 4 gives a 24-layer structure while crystallized. Using a bumpy bottom, the crystallized state and disordered state are distinct and nearly uniform for the entire packing, as different final states of the evolution. For thickness below about 15 layers, the distinction between the two states becomes less well-defined, and the crystallized state becomes “quantized” with respect to the exact filling. For packings thicker than 30 layers, stationary sheared flows with a non-uniform degree of order can be created and persist indefinitely, depending on the building history of the packing. (2) The anomalous mobility observed at the reversal of shearing exists only when the thickness exceeds about 13 layers, as shown in Fig. 5.13. (3) The
localization of the velocity field (the shear banding) requires sufficient thickness. The velocity profile approaches a linear profile from top to bottom when the packing approaches the thin limit, as is demonstrated by the 5-layer flow shown in Fig. 5.9. Note that as the aspect ratio $H_0/W_0$ decreases, the velocity field ceases to reflect the properties of the materials in the channel because it always approaches a linear profile - for instance, compare the hypothetical profiles of ordinary fluid flow at small $H_0/W_0$ as shown in Fig. A.2.

Our precision measurement of the strongly non-linear velocity profile, where the local velocity can exhibit a smooth change by several decades from top to bottom, raises significant challenges for theoretical modeling. First, while the states of internal order show a decisive impact on the vertical decay of particle velocity, changing the particle size by about a factor of three shows that the resultant velocity fields have similar initial slope with respect to physical distance $z$ on the semi-log plot, but not with respect to the dimensionless coordinates $z/d$ (Fig. 5.4). Therefore, while the internal order clearly affects the local rheology of the material, the role of particle size $d$ is unclear. On the other hand, by progressively increasing the layer thickness of flows using the same particle size, the velocity profile deviates from linear relation but eventually collapse onto one master curve that is not influenced by either the surface condition or the location of the bottom boundary $(-H_0)$. This fact seems to suggest that the drag exerted by the sidewalls plays an important role in determining the vertical decay of velocity, even though one might have expected the observed length scale of vertical decay to be comparable to the channel width. (The decay towards the two sidewalls in the horizontal velocity profile at a fixed height [Fig. 5.1] qualitatively demonstrates this effect, although the effect is explicit only for the disordered state.) In Section 5.3.5, a heuristic local model with no explicit reference to $d$ is attempted to reproduce an exponential like vertical decay of the velocity field (the shear banding) by considering the frictional drag from the sidewalls: based on this simplistic model, the resultant vertical decay length is proportional to the channel width $W_0$, but can be much smaller than $W_0$ as long as the local shear stress is only weakly dependent on shear rate.

As is discussed in Section 5.3.2 and 5.3.1, the shear banding and the velocity distribution of packed granular particles often depends on the global geometry of the system. To test theoretical ideas against experimental measurements, it is necessary to implement the
theory in a 3D context with explicit assumptions of boundary conditions.

The experimental results in this dissertation raise several theoretical questions. For instance, the observation of shear-induced crystallization and the history-dependent multiple routes of evolution point to the following issues: how to relate the granular rheology to the internal structure; how to explain the ordering mechanisms under steady or oscillatory driving; how to construct a model with a minimal set of internal state variables to account for the stability of different internal structures, the apparent stochastic selection of the final state, and the shear-induced stabilization. In Section 4.3, I speculate about the possibility of constructing a dynamical systems picture of the state of order of the system. I also discuss in Section 5.3.4 the view of seeing the quasi-static dynamics as rate-independent adjustments of grains subject to imposed constraints. Since long-term evolution is often observed in other physical systems such as glassy or nearly-jammed materials, comparison to these evolving systems can be interesting. The question of length scale is also challenging. What is the role of grain size $d$ in the global ordering and local rheology? More generally, do local models, where the state of stress only depends on local variables (see Ref. [59, 5] for examples), suffice for describing dense granular flows, or are non-local approaches [9, 12] necessary?

On the other hand, the connection between quasi-static flows and granular static states [14] is also interesting: given the observed irreversible evolution with time, our observations raise the question of whether a slowly creeping flow can be viewed as a stationary ergodic ensemble of all possible static particle configurations (as is commonly assumed, for instance, in Ref. [32]). It may instead be a weighted subset in which some configurations of particles become inaccessible as the total volume evolves. To understand the gradual evolution of slowly evolving granular materials, further development of experimental probes of the evolving internal structure, supplemented by non-invasive detection of inter-particle stress (see, e.g. Ref. [24]), seems highly desirable.

Finally, examples of future work or possible extensions are listed here:

- Further experiments are under way to understand other factors that influence the time evolution of packed granular material under shear, for instance, the role of particle anisotropy or polydispersity, which leads to size segregation and should suppress
crystallization. This investigation should lead to the understanding of granular flows in more realistic circumstances.

- Continuous sampling of granular volume as the sheared packing evolves in time, for instance, the successive sinking of the upper boundary in response to shear reversals and the gradual recovery (dilation) when an unidirectional shearing is resumed, can be used as a direct test of theoretical predictions or computer simulations of contact dynamics.

- Using particles smaller by an order of magnitude may provide useful information regarding the role of particle size in determining velocity field. Does the flow field exhibit a fluid limit where the particle size is irrelevant? How does this limiting field, if exists, compare to the internal velocity fields obtained in this dissertation work? In addition, by shrinking the particle size, the notion of local dynamics may be justified as the typical length of force chains becomes less than the channel width.

- The onset of motion as stress is gradually increased, may worth investigating. It may be related perhaps to the process of fracture in materials.
Appendix A

Theoretical treatments
A.1 Criterion on stress and shear rate for creating a quasi-static granular flow

In this appendix, we derive a general criterion for the stress and shear rate for granular particles to be in a quasi-static regime. We use this criterion to estimate the upper bound of driving speeds for the sheared packing to be entirely in the state of quasi-static (creeping) flow, under our fixed normal load.

Consider uniformly sized, non-cohesive, rigid particles of mass $m$ and diameter $d$, packed under a compressive load along $z$ and sheared along $+x$ and $-x$, as shown in Fig. A.1. Here we choose a local reference frame in which the value of time-averaged velocity $\langle V_x \rangle$ as a function of height $z$ appears anti-symmetric with respect to $z_0$, a reference height where the two highlighted particles contact. If the local granular stress is high enough to keep all contacting particles in either static or sliding contact for a finite duration (until they are eventually pulled apart), these two particles would move around the contact point with a radius of curvature $d/2$ and an angular velocity $\omega \approx \langle V_x \rangle_{z_0+0.5d}/0.5d \equiv \dot{\gamma}$. Because the assumed non-cohesive contacts between the two spheres only sustain a positive stress but not a tension, this is equivalent to the condition that the ‘normal’ component
of the total force exerted by all surrounding particles on each of the two highlighted spheres needs to be greater than the centripetal force \( m\omega^2d/2 \) required for the local circular motion. That is,

\[
m\dot{\gamma}^2d/2 \approx m\omega^2d/2 < F_N \approx \sigma_{zz}d^2 \tag{A.1}
\]

in which \( \sigma_{zz} \) stands for coarse-grained normal stress along the \( z \) direction. We can define a dimensionless number \( X \equiv \frac{1}{2}m\dot{\gamma}^2d^{-1}\sigma_{zz}^{-1} \), which represents \(^1\) the ratio of the centripetal force per particle to the normal force per area \( d^2 \). The stress component \( \sigma_{zz} \) can either be mechanically imposed, or arise from grain’s self-weight, or both. We can alternatively express \( \sigma_{zz}d^2 \) as \( mgN_{eq} \), in which \( N_{eq} \) is the equivalent number of layers whose weight produces a stress \( \sigma_{zz} \) at the location being considered. For granular materials in a sufficiently wide container where the Janssen effect can be neglected \([38]\), grain weight contributes to the normal stress at any horizontal internal plane with the corresponding \( N_{eq} \) equal to the actual number of layers packed above the plane considered.

The shear rate \( \dot{\gamma} \) can be approximated by \( \langle V_x \rangle /l \) where \( l \) is the local exponential decay length for grain velocity. The empirical value of \( l \) is usually about a few particle diameters. For a packing sheared from above under the influence of gravity, the highest shear rate \( \dot{\gamma} \) and lowest stress \( \sigma_{zz} \) occur at the first layer adjacent to the shearing surface; therefore the criterion Eqn (A.1) becomes

\[
m\left(\frac{U_0}{l}\right)^2d/2 < \sigma_{zz}d^2 \tag{A.2}
\]

or alternatively

\[
m\left(\frac{U_0}{l}\right)^2d/2 < mgN_{eq} \tag{A.3}
\]

where \( U_0 \) stands for the mean velocity of the first layer, which is close to the boundary speed. Using the parameters \( d = 10^{-1}\) cm, \( N_{eq} = 10^2 \), \( g = 10^3\) cm/s\(^2\) and an empirical value \( l = 3d \), we estimate that the transition threshold for \( U_0 \) is \( 4 \times 10^3d/s \), which is more than two orders of magnitude higher than the highest driving speed \( 12d/s \) used in our experiments. Therefore, the shear flows reported in this experimental work are entirely in the quasi-static (creeping) regime where \( X << 1 \).

\(^1\)One can perhaps compare this dimensionless number with the Mach number defined in kinetic theories. With the substitutions \( v = \dot{\gamma}d, P_0 = \sigma_{zz}, \rho_0 \sim md^{-3}, \) and the definition (Mach number) \( \equiv v/(\gamma P_0/\rho_0) \), one finds that \( X \sim (\text{Mach number})^2 \). (The \( \gamma \) here stands for the constant for adiabatic expansion \( D(P\rho^{-\gamma})/Dt = 0 \), with a value of \( \approx 1.4 \) for air.)
When the shear rate $\dot{\gamma}$ is high or the stress is low, it is possible to reverse Eqn (A.1) and the subsequent inequalities; for example, transition from the regime of creeping flow into the ‘granular gas’ regime can be found either locally or globally in high-speed annular shear cells [19, 25] with their normal load reduced by a counter-weight. For an inclined gravity-driven granular flow with an upper free surface, it is commonly observed that particles within a few grain diameters of the surface are in a dynamical regime that is qualitatively different from that of the creeping grains in the interior. (See the experiments and measured velocity profiles in Ref. [27, 50].) Note that with an upper free surface, the smallest meaningful value of $N_{eq}$ in the estimate proposed here should be 1 instead of 0, due to the weight of the first layer itself.

In the preceding discussion, cohesivity, friction, finite rigidity (elasticity) of grains and their interaction with interstitial fluids are neglected; the analysis here is meant to be an order-of-magnitude estimate of the threshold above which the departure from creeping flow can occur.

A.2 Velocity fields of ordinary fluid in a rectangular channel

In this appendix, we analyze the velocity field if the granular material in the channel is replaced with a fluid. The momentum balance for a continuous medium of density $\rho$ is in general

$$\rho \frac{D}{Dt} v_i = \partial_j \{T_{ij}\} + \rho g_i \tag{A.4}$$

where $\frac{D}{Dt} \equiv (\frac{\partial}{\partial t} + \vec{v} \cdot \nabla)$ and $\partial_j \equiv \frac{\partial}{\partial x_j}$. For an isotropic fluid, the stress $T_{ij}$ has the form

$$T_{ij} = -p \cdot \delta_{ij} + \eta(\partial_j v_i + \partial_i v_j) \tag{A.5}$$

where $p$ stands for pressure. For a Newtonian fluid, the viscosity $\eta$ is by definition a constant independent of the local velocity gradient of the fluid.

Assuming the fluid is incompressible, i.e. $\nabla \cdot \vec{v} = 0$, Eqn (A.4) reduces to

$$\rho \frac{D}{Dt} v_i = (\partial_j \eta)(\partial_j v_i + \partial_i v_j) + \eta \nabla^2 v_i - \partial_i p + \rho g_i \tag{A.6}$$

which is in the form of the Navier-Stokes equation except for the additional non-Newtonian term $(\partial_j \eta)(\partial_j v_i + \partial_i v_j)$.

\footnote{For simplicity, the distinction between shear viscosity and extensional viscosity is ignored.}
Figure A.2: The solution of Laplace equation $\nabla^2 \Phi = 0$ with no-slip boundary conditions: $\Phi|_{z=0} = 1$ for the upper boundary, $\Phi|_{y=0} = \Phi|_{y=W_0} = 0$ for the two vertical sidewalls, and $\Phi|_{z=-H_0} = 0$ for the bottom. (a) Vertical profiles of $\Phi$ at $y = W_0/2$ as functions of $z$, for different values of $H_0$. The curves approach a linear profile when the aspect ratio $\frac{H_0}{W_0}$ is small, and an exponential profile when $\frac{H_0}{W_0}$ is large. (b) Horizontal profiles of $\Phi$ at the mid-height $z = -H_0/2$ as functions of $y$, for different values of $H_0$. 
Assuming the stationary solution has the form \( \vec{v} = \Phi(y,z)e_x \) with the gravity \( \vec{g} \) being parallel to the \( z \) axis, Eqn (A.6) reduces to

\[
0 = (\nabla_{yz}\eta) \cdot (\nabla_{yz}\Phi) + \eta \nabla^2_{yz}\Phi \quad (A.7)
\]

In the simplest case of a Newtonian fluid where \( \eta \) is a constant, the problem reduces to Laplace equation \( \nabla^2_{yz}\Phi = 0 \). By specifying boundary conditions \( \Phi|_{y=0} = \Phi|_{y=W_0} = 0 \) for the two vertical sidewalls, \( \Phi|_{z=-H_0} = 0 \) for the bottom, and a flat profile \( \Phi|_{z=0} = 1 \) at the upper surface, the solution is

\[
\Phi(y,z) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \left( \frac{\sinh \lambda_n z \tanh \lambda_n H_0 + \cosh \lambda_n z}{\lambda_n} \right) \sin \frac{n\pi y}{W_0} \quad (A.8)
\]

in which \( \lambda_n \equiv n\pi/W_0 \). The principal (longest) decay length in this case is \( \lambda_1^{-1} = W_0/\pi \), which is also reflected by the approximate slope of the curves shown in the semi-log plot Fig. 5.5. Samples of vertical and horizontal profiles of Eqn (A.8) for our channel with a fixed width \( W_0 \), at different packing thickness \( H_0 \), are demonstrated in Fig. A.2.

Note that, more generally, if one specifies an arbitrary velocity profile at \( z = 0 \) with no-slip conditions on the other three stationary boundaries, the principal decay length will be selected by the lowest non-vanishing wave number along \( y \) as \( \Phi|_{z=0} \) is decomposed into sinusoidal components. That is, the principal length is determined by the spatial symmetry of the speed profile at \( z = 0 \).

### A.3 Derivation of Eqn (5.7)

By introducing a velocity-dependent sidewall friction Eqn (5.6) and assuming \( v^{\text{slip}}(z) = V_x(z) \), the stress balance Eqn (5.2) yields a differential-integral equation

\[
\ln[V'_x(z)/V'_x(0)] = \frac{1}{L_\alpha} \int_0^z (1 + \beta \ln[V_x(z')/V_x(0)])dz' \quad (A.9)
\]

in which \( L_\alpha \equiv \left( \frac{2w_0}{\sigma \alpha W_0} \right)^{-1} \). This mathematical problem can be defined alternatively in terms of the local decay rate \( \lambda(z) \equiv V'_x(z)/V_x(z) \). Using the identity \( V'_x(z) = \lambda(z)V_x(z) = \lambda(z)V_x(z) \exp \int_0^z \lambda(z')dz' \), Eqn (A.9) can be rewritten as

\[
\ln[\lambda(z)/\lambda(0)] + \int_0^z \lambda(z')dz' = \frac{1}{L_\alpha} \int_0^z (1 + \beta \int_0^{z'} \lambda(z'')dz'')dz' \quad (A.10)
\]
and $V_x(z)$ can be obtained after $\lambda(z)$ is determined. Differentiating Eqn (A.10) gives

$$\lambda(z) = -\frac{\lambda'(z)}{\lambda(z)} + \frac{1}{L_\alpha}(1 + \beta \int_0^z \lambda(z')dz')$$

(A.11)

which is a differential-integral equation of $\lambda(z)$.

In order to find approximate solutions to Eqn (A.10), we use this equation to define an iteration. We denote the n-th iteration as $\lambda^{(n)}(z)$ and assign $\lambda^{(0)}(z)$ to be a constant. First of all,

$$\lambda^{(0)}(z) = \frac{1}{L_\alpha}$$

(A.12)

is required by the self-consistency at $z = 0$. Iterating Eqn (A.10) gives

$$\lambda^{(1)}(z) = \frac{1}{L_\alpha}(1 + \beta L_\alpha z)$$

(A.13)

$$\lambda^{(2)}(z) = -\frac{\beta}{L_\alpha} \frac{1}{1 + \frac{\beta}{L_\alpha}z} + \frac{1}{L_\alpha}(1 + \beta L_\alpha(z + \frac{\beta}{2L_\alpha}z^2))$$

(A.14)

and successively higher order terms of $\frac{\beta}{L_\alpha}$.

By integrating the first approximation Eqn (A.13), we obtain

$$\ln\left[\frac{V_x(z)}{V_x(0)}\right] = \frac{1}{L_\alpha} \cdot (z + \frac{\beta}{2L_\alpha}z^2)$$

(A.15)

which has the form of a quadratic function that satisfactorily fits the experimental data described in Fig. 5.4.

It should be pointed out that neither the iteration procedure is unique, nor the iteration is the only method to approximate the solution. The convergence and effectiveness of different approximation schemes that can be used to solve Eqn (A.10) need further investigation. Any useful approximation scheme or numerical solution may require a validity internal in $z$ that is several times wide the assumed value of $L_\alpha$. 

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Appendix B

Supplements
B.1 Tracking individual particles

B.1.1 Quality of Dark Disk Tracking

The algorithm for the Dark Disk Tracking is defined in Section 2.2 and is implemented using Interactive Data Language (IDL).

Fig. B.1 shows the quality of this automated particle recognition. The assumed apparent radius $a_0$ described in Section 2.2 is 22 pixels for (a-c) and 9 pixels for (d); the widths of the Gaussian ring $\sigma_0$ are 2 pixels and 1 pixel respectively. The algorithm for particle recognition is in the spirit of edge detection, and can therefore tolerate certain degrees of the apparent overlapping of particles. Errors due to the imperfection of the apparent shape of particles and the resolution limit set by the pixel width can be estimated from these graphs. In addition, the light sheet currently created by the Argon laser has a non-uniform intensity profile that fluctuates randomly with time (as seen from the sequential images of a static packing), and causes random shifts of the computed particle centers at the order of 2 pixels.

B.1.2 Statistics on measured individual particle displacements

Examples of the statistics of individual particle displacements are shown in Fig. B.2. The statistics are based on the trajectories computed from sequential images at the magnification identical to that of Fig. B.1(a). The sheared packing is in a disordered state, with the illumination light sheet being vertical (in the $xz$ plane). A positive $\Delta z$ represents a displacement upwards.

Fig. B.2(a) shows the relative probability of the vertical displacements $\Delta z$ of all recognized particles in the field of view during a fixed time interval $\Delta t$, where $\Delta t \approx \dot{\gamma}_{\text{local}}^{-1}$. ($\dot{\gamma}_{\text{local}}$ stands for the local shear rate.) The statistics are based on particle positions extracted from $10^4$ images. One interesting feature of this statistics is the lack of the mirror symmetry with respect to $\Delta z = 0$. The asymmetry may be a combined result of the highly non-uniform velocity gradient (the shear banding), and the fact that gravity plays a role in some rare events during which a particle temporarily lose contacts from multiple

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1. The source code of the main program is self-documented and the current version is named as jc_DDTrack_c07.pro; the main program also calls other subroutines that are described therein.
2. Throughout the parameter range of the quasi-statically sheared packing studied in this dissertation, the probability of the events of the void filling by isolated individual particles is at the order of $10^{-3}$. This
Figure B.1: Sample images showing the quality of Dark Disk Tracking. In all graphs, crosses are overlaid on an image, to indicate particle centers as determined by the computer routines. The square grids in black mark the blocks of 50 pixels in either direction. (a) The original image overlaid with the final results. (b) The convolution (defined in Section 2.2) of the original image. The locations of local maximums are used as the first approximations in determining the centers of individual particles. Some of the local maximums are discarded by imposing a threshold in the sharpness of the bright peaks. (c) The superposition of the original image (in green), the convolution (in red), and the crosses showing the final results (in blue). (d) Another sample with a different magnification, and with a lower threshold to admit more recognized particles (at the cost of the accuracy of the their positions).
Figure B.2: Statistics of captured vertical displacements of individual particles. The width of one pixel is approximately $\frac{1}{40}$ of the particle diameter. (a) The distribution of detected vertical displacements $\Delta z$ during a fixed time interval $\Delta t$. The distribution shows a noticeable asymmetry despite the possible statistical bias due to the light sheet illumination. (b) The apparent root-mean-square vertical displacement as a function of time interval $\Delta t$. The noise floor is an estimate based on random errors described in Section B.1.1.
neighbors and fall or roll into a void.

It is important to note that, because the recorded lengths of individual particle trajectories are partly determined by the durations of particles staying in the thickness of the light sheet, the statistics of the individual $\Delta z$ can be substantially biased, particularly when the tails of the this probability distribution function (PDF) involve displacements comparable to the particle radius. Therefore the PDF shown in Fig. B.2(a) may not represent the true PDF of $\Delta z$. The asymmetry of the distribution, nevertheless, should still be a valid result in the presence of this bias.

When $\dot{\gamma}_{\text{local}} \Delta t$ is kept fixed, the shape of the measured PDF appears invariant for the change of $\dot{\gamma}_{\text{local}}$ by a few orders of magnitude (in measurements at different locations and with different driving speeds). This is consistent with the picture that $\dot{\gamma}_{\text{local}}$ is perhaps the only relevant parameter that determines the local time scale for the quasi-statically sheared packing. The invariance does not require a strictly unbiased sampling of $\Delta z$, either.

Fig. B.2(b) shows the root-mean-square vertical displacement during different time interval $\Delta t$. However, the dynamical range of this measurement is very limited, because of the sampling bias mentioned previously, and of the 2-pixel random noise described in Section B.1.2. The apparent crossover of the slope is inconclusive.

### B.2 Precision measurements of velocity fields

#### B.2.1 Local imaging at different frame rates and image calibration

To determine the highly non-linear velocity field with a dynamical range of several orders of magnitude, we first measure the local velocity profiles obtained from sequential images recorded at different frame rates for different portions of the sheared packing. Fig. B.3 demonstrates the frame rates for local image patches used in a typical experiment of a 24-layer flow.

The absolute calibration of images (the conversion of pixel width to physical distance) is crucial for combining the local segments of velocity profiles correctly. The principle for this calibration procedure is to physically translate the camera by a known distance parallel to the imaging plane, and compare the translation with the corresponding shift of images in estimate is based on examining high-frame-rate imaging with a $\Delta t \sim 0.001 \ddot{\gamma}^{-1}_{\text{local}}$.  

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Figure B.3: The sampling rates for different local image patches that are used to determine the vertical profile of a 24-layer crystallized packing described in Fig. 5.4. The crystallized state has reached its asymptotic velocity profile after the continuous shearing at a speed of $12d/s$ for a few days. For each segment (a few particle diameters wide along $z$) of the profile, 2084 frames are recorded using the appropriate sampling rate listed on the graph.

pixels. One convenient way to do so is to first place a ruler (with lines marking millimeters, for example) in front of the transparent glass wall and keep the particles static. Before translating the camera, record an image with the focus on the intended internal plane, then change the focus of the image to the ruler and record one image. Afterwards, translate the camera (by raising or lowering its height set by a tripod) for a distance that can be determined by monitoring the apparent translation the ruler in the image, then restore the focus of the image to the intended internal plane and record the static image again. The apparent shift of the ruler image gives the physical translation, while the number of pixels (on the imaging plane) that corresponds to this physical distance is determined by comparing the two internal slice images. The pixel-distance conversion can be determined in an accuracy of one percent in this procedure.
B.2.2 Stochastic variation and measurement uncertainties

Fig. B.4 summarizes the stochastic variation and measurement uncertainties in measuring the vertical profile of the mean velocity field, using the typical 24-layer crystallized flow as an example.

Each segment of the velocity profile is determined from a recording of \( N = 50 \) to 500 particles passing by each statistical bin (about 1\( d \) in \( z \)) that corresponds to one row of particles. The uncertainty due to the errors of particle tracking can be estimated by \( \delta_{\text{pixel}} \approx \frac{e}{L \sqrt{N}} < 0.01 \), where the pixel error \( e < \frac{d}{10} \), the typical length of trajectories \( L \sim \) a few times of \( d \), and \( N > 50 \).

For a 24-layers crystallized flow at its asymptotic state, the value of the finite-time average of the local velocity fluctuates around the true mean. This uncertainty is labeled as \( \delta \) and can be estimated by multiple measurements of the local mean velocity at the same location in the course of non-stop shearing (for days). With the typical recording length \( (N = 50 \) to 500), the fractional fluctuation around the true mean is at the order of 30% and is roughly uniform throughout the entire packing.

There are stochastic variations of the final velocity profiles between different experiments. Different experiments are separated by fully stirring the beads or refilling the cell. The two data sets demonstrates the largest deviation between different experiments ever recorded. At the lower part of the flow, the variation \( \Delta \) can be as large as a factor of 3, but is still smaller than the typical difference between the profiles of the crystallized and disordered state. This stochastic variation may reflect the density of defects that are ‘frozen’ in the packing during the course of crystallization.
Figure B.4: Variation and uncertainties in measuring the vertical profile of time-averaged velocity of a 24-layer crystallized flow in a stationary state. The imaging error $\delta_{\text{pixel}}$ is negligible compared to $\delta$, the fluctuation of measured value around the true mean, which is in turn small compared to the largest stochastic variation ($\Delta$) between two 'different experiments' (Data 021015 and Data 021109) as described in text.
Bibliography


