Stability of $n$-covered circles for elastic rods with constant planar intrinsic curvature

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Abstract

A stability index is computed for the $n$-covered circular equilibria of inextensible-unshearable elastic rods with constant planar intrinsic curvature $\theta$ and constant values for the twisting stiffness and two bending stiffnesses. A simple expression is derived for the index as a function of $\theta, \rho$ (the ratio of bending stiffness out of the plane of curvature to bending stiffness in the plane of curvature), and $\gamma$ (the ratio of twisting stiffness to bending stiffness in the plane of curvature). In particular, for intrinsically straight rods ($\theta = 0$) we prove that the 1-covered circle is stable if and only if $\rho \geq 1$, and the $n$-covered circle ($n > 1$) is stable if and only if $\gamma > 1, \rho > 1$, and $\frac{n-1}{n} \leq \sqrt{\frac{\gamma - 1}{\rho}}$.

The index is computed by framing the standard Euler-Camot equilibrium equations within a constrained variational principle with an isoperimetric constraint ensuring the ring closure. The fact that $\theta$ appears linearly in the second variation allows the second variation to be diagonalized using the eigenvectors of an appropriate eigenvalue problem similar to a Sturm-Liouville problem. This diagonalization allows the direct computation of an unconstrained index (disregarding ring closure). We then apply a result of Maddocks [8] to find the constrained index in terms of this unconstrained index and a correction computable from the linearized constraint.

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1 Introduction

In this paper, we study the stability of untwisted circular equilibria for an intrinsically curved, anisotropic, elastic rod. These equilibria are formulated as critical points of a strain energy functional $E$ in an isoperimetrically constrained calculus of variations problem. Following standard practice [5, 8], we define a constrained index $\mathcal{I}$ to be the dimension of a maximal subspace of allowed variations on which the second variation $\mathcal{S}$ of $E$ is negative. We call an equilibrium stable if its constrained index is zero; this is a necessary condition for the equilibrium to be a constrained local minimum of $E$.

Maddocks [8] provided a formula for the constrained index in terms of an unconstrained index $\mathcal{I}_u$ and the number of nonpositive eigenvalues $d^{np}(W)$ of a particular matrix $W$. Each of these quantities can be computed in closed form for the circular equilibria considered here. Particular use is made of the fact that the intrinsic curvature parameter $\hat{u}$ appears linearly in $\mathcal{S}$. Because of this, the eigenvectors $\zeta$ of the Sturm-Liouville-like eigenvalue problem

$$\mathcal{S}(\hat{u})\zeta = 0, \quad \zeta(0) = \zeta(1) = 0$$

can be used to construct a diagonalizing basis for $\mathcal{S}(\hat{u})$ at each fixed $\hat{u}$, thereby allowing a direct computation of $\mathcal{I}_u$.

These computations provide an explicit formula for the dependence of the constrained index $\mathcal{I}$ on the rod parameters. For example, we show that for each positive integer $n$ and set of rod stiffnesses, there is an interval of values for the rod intrinsic curvature $\hat{u}$ on which the $n$-covered circular configuration is stable. As a particular application, we determine the values for the stiffnesses at which an intrinsically straight anisotropic rod yields stable $n$-covered circles.

The remainder of this paper is organized as follows. The variational formulation of the equations for circular equilibria is presented in Section 2, and the appropriate second variation is computed in Section 3. The index formula of Maddocks is given in Section 4, and the hypotheses needed to apply the formula are verified. The general solution of $\mathcal{S}\zeta = 0$ is computed in Section 5, then used in Sections 6–7 to compute $\mathcal{I}_u$ and $W$. Finally, results are presented in Section 8.

2 Equilibrium equations for an elastic rod

Here we summarize the standard Cosserat theory of inextensible and unshearable elastic rods, as surveyed, for example, in [1]. The configuration of a rod is described by a centerline $r(s)$ (written as a function of arclength $s$) and a set of directors $\{d_1(s), d_2(s), d_3(s)\}$ that form an orthonormal frame giving the orientation of the cross-section of the rod. For convenience, we choose a length scale so that $0 \leq s \leq 1$. The assumption of inextensibility and unshearability of the rod is incorporated in the requirement that $d_3(s)$, the director orthogonal to the rod cross-section, equals $r'(s)$, the tangent vector to the centerline.
Orthonormality of the directors implies the existence of a (Darboux) vector \( \mathbf{u}(s) \) defined by the kinematic relations:

\[
\mathbf{d}_i'(s) = \mathbf{u}(s) \times \mathbf{d}_i(s), \quad i = 1, 2, 3.
\]

The components of \( \mathbf{u} \) in the rod frame are denoted by \( u_i(s) = \mathbf{u}(s) \cdot \mathbf{d}_i(s) \) and are called the strains.

It will be convenient to describe the directors via Euler parameters or quaternions \( \mathbf{q} \in \mathbb{R}^4 \), since then the directors and strains are given by rational functions:

\[
\begin{align*}
\mathbf{d}_1 &= \frac{1}{|\mathbf{q}|^2} \begin{bmatrix}
q_1^2 - q_2^2 - q_3^2 + q_4^2 \\
2q_1q_2 + 2q_3q_4 \\
2q_1q_3 - 2q_2q_4
\end{bmatrix}, \\
\mathbf{d}_2 &= \frac{1}{|\mathbf{q}|^2} \begin{bmatrix}
-2q_2q_3 - 2q_1q_4 \\
q_1^2 + q_2^2 - q_3^2 + q_4^2 \\
2q_1q_3 + 2q_2q_4
\end{bmatrix}, \\
\mathbf{d}_3 &= \frac{1}{|\mathbf{q}|^2} \begin{bmatrix}
2q_1q_3 + 2q_2q_4 \\
q_1^2 - q_2^2 + q_3^2 + q_4^2 \\
-2q_2q_3 - 2q_1q_4
\end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
u_i &= \frac{2}{|\mathbf{q}|^2} \mathbf{B}_i \mathbf{q},
\end{align*}
\]

where

\[
\begin{align*}
\mathbf{B}_1 &= \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}, \\
\mathbf{B}_2 &= \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}, \\
\mathbf{B}_3 &= \begin{bmatrix}
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}.
\end{align*}
\]

We impose the following ring constraints on the elastic rod:

\[
\int_0^1 \mathbf{d}_3(s) \, ds = 0, \quad \mathbf{q}(0) = (0, 0, 0, 1), \quad \mathbf{q}(1) = (0, 0, 0, \pm 1).
\]

The first constraint implies, due to the inextensibility-unshearability condition \( \mathbf{r}'(s) = \mathbf{d}_3(s) \), that \( \mathbf{r}(0) = \mathbf{r}(1) \), i.e., that the rod begin and end at the same point. The last two constraints imply that the rod directors are lined up with the coordinate axes at \( s = 0 \) and \( s = 1 \) (the \( \pm \) freedom in \( \mathbf{q}(1) \) is due to the double-covering of the space of directors by the unit quaternions; it will not play an important role here). For convenience, we rewrite the vector integral constraint \( \int_0^1 \mathbf{d}_3(s) \, ds = 0 \) as three scalar integral constraints:

\[
\int_0^1 d_{3i}(s) \, ds = 0,
\]

where \( d_{3i} \) is the \( i \)-th component of the vector \( \mathbf{d}_3 \) in Eq. (1).

The elastic energy of the rod is expressed in terms of the strains, and we assume here a commonly used quadratic energy:

\[
E[\mathbf{q}] = \int_0^1 \sum_{i=1}^3 \left[ \frac{1}{2} K_i(s) \left[ u_i(\mathbf{q}(s), \mathbf{q}'(s)) - \dot{\alpha}_i(s) \right]^2 \right] \, ds
\]
where \( K_i(s) \) are the bending \((i = 1, 2)\) and twisting \((i = 3)\) stiffnesses of the rod, and \( \hat{u}_i(s) \) are intrinsic strains describing the intrinsic bending \((i = 1, 2)\) and twisting \((i = 3)\) of the rod. We will assume that the stiffnesses and intrinsic strains are independent of \( s \), and that \((\hat{u}_1, \hat{u}_2, \hat{u}_3) = (\hat{u}, 0, 0)\), i.e., constant planar intrinsic bending and zero intrinsic twist. We focus here on equilibria of the rod energy (3) subject to ring constraints; i.e., we consider the variational problem to find functions \( \mathbf{q}(s) \) at which \( L[\mathbf{q}] \) is stationary subject to the constraints (2). A standard necessary condition for this stationarity is the Euler-Lagrange equation

\[
\frac{d}{ds} (L\mathbf{q}) = L_{\mathbf{q}},
\]

of the extended functional

\[
J[\mathbf{q}] \equiv \int_0^1 L(\mathbf{q}(s), \mathbf{q}'(s)) \, ds \equiv \int_0^1 \sum_{i=1}^3 \left[ \frac{1}{2} K_i [u_i(\mathbf{q}(s), \mathbf{q}'(s)) - \hat{u}_i]^2 + d_{3i}(\mathbf{q}(s)) \lambda_i \right] \, ds,
\]

with Lagrange multipliers \( \lambda_i \) included to account for the isoperimetric constraints \( \int_{0}^{1} d_{3i} \, ds = 0 \). Here, subscripting by \( \mathbf{q} \) or \( \mathbf{q}' \) indicates partial differentiation with respect to the indicated variable. For this particular functional \( J \), the Euler-Lagrange equations are:

\[
\frac{d}{ds} \sum_{i=1}^3 \left[ K_i(u_i - \hat{u}_i)^2 |\mathbf{q}|^2 \right] = \sum_{i=1}^3 \left[ K_i(u_i - \hat{u}_i) \frac{|\mathbf{q}|^2 (-2B_i q' - 2(q')^T B_i q)}{|\mathbf{q}|^4} \right] + \sum_{i=1}^3 (d_{3i})_{q} \lambda_i.
\]

(4)

Recall that we take \((\hat{u}_1, \hat{u}_2, \hat{u}_3) = (\hat{u}, 0, 0)\).

For each positive integer \( n \), we consider the \( n \)-covered circular equilibria:

\[
\mathbf{q}(s) = \begin{bmatrix} \sin(n\pi s) \\ 0 \\ \cos(n\pi s) \end{bmatrix}, \quad \lambda_1 = \lambda_2 = \lambda_3 = 0.
\]

(5)

These are readily verified to be solutions to (4) for any value of \( \hat{u} \). The goal of this paper is to assign to these \( n \)-covered circles a stability index, in the sense defined in the following section.

### 3 The second variation

Throughout this article, given two functions \( f, g : [0, 1] \to \mathbb{R}^3 \), we adopt the following notation for their \( L^2 \) inner product:

\[
\langle f, g \rangle \equiv \int_0^1 f(s)^T g(s) \, ds.
\]

Given an equilibrium \( \mathbf{q}^{eq} \), we define an allowed variation to be any \( \delta \mathbf{q} \) so that \( \mathbf{q}^{eq} + \delta \mathbf{q} \) satisfies the ring constraints (2) to first order in \( \delta \mathbf{q} \), i.e., \( \delta \mathbf{q}(0) = \delta \mathbf{q}(1) = 0 \) and \( \langle (d_{3i})_{q}^{eq}, \delta \mathbf{q} \rangle = 0 \) for \( i = 1, 2, 3 \), where the superscript \( eq \) indicates evaluation at the equilibrium.
The index of an equilibrium is defined in terms of the second variation of \( J \):

\[
\delta^2 J[\delta \mathbf{q}] = \int_0^1 \left[ (\delta \mathbf{q})^T L_{qq}^{eq} \delta \mathbf{q} + (\delta \mathbf{q}')^T L_{qq}'^{eq} \delta \mathbf{q} + (\delta \mathbf{q})^T L_{qq}^{eq} \delta \mathbf{q}' + (\delta \mathbf{q}')^T L_{qq}'^{eq} \delta \mathbf{q}' \right] \, ds,
\]

where \( \delta \mathbf{q} \) is any allowed variation. We define the index \( \mathcal{I} \) of the equilibrium to be the maximal dimension of a subspace of allowed variations on which \( \delta^2 J < 0 \).

### 3.1 Accounting for the scale invariance in the quaternions

We note first a property of \( \delta^2 J \) peculiar to the example at hand. The integrand \( L \) is invariant to a scaling of \( \mathbf{q} \), i.e.,

\[
L(\alpha \mathbf{q}, \alpha \mathbf{q}') = L(\mathbf{q}, \mathbf{q}')
\]

for any \( \alpha \) (this degeneracy arises from our use of four-dimensional quaternions to represent the three-dimensional space \( SO(3) \) of directors). This implies that the integrand of \( \delta^2 J \) vanishes when \( \delta \mathbf{q} = \mathbf{q}^{eq} \), as may be verified by differentiating Equation (7) twice with respect to \( \alpha \) and setting \( \alpha = 1 \) and \( \mathbf{q} = \mathbf{q}^{eq} \). Therefore, if at each \( s \) we expand any allowed variation \( \delta \mathbf{q}(s) \) in an orthonormal basis \( \{ \mathbf{v}_1(s), \mathbf{v}_2(s), \mathbf{v}_3(s), \mathbf{q}^{eq}(s) \} \) of \( \mathbb{R}^4 \):

\[
\delta \mathbf{q}(s) = \zeta_1(s) \mathbf{v}_1(s) + \zeta_2(s) \mathbf{v}_2(s) + \zeta_3(s) \mathbf{v}_3(s) + \beta \mathbf{q}^{eq}(s),
\]

then \( \delta^2 J[\delta \mathbf{q}] = \delta^2 J[\zeta_1 \mathbf{v}_1 + \zeta_2 \mathbf{v}_2 + \zeta_3 \mathbf{v}_3] \). For example, if we define

\[
\Pi(s) \equiv \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & \cos(n \pi s) & 0 \\ \cos(n \pi s) & 0 & -\sin(n \pi s) \\ \sin(n \pi s) & 0 & \cos(n \pi s) \\ 0 & -\sin(n \pi s) & 0 \end{bmatrix},
\]

and write an arbitrary \( \delta \mathbf{q}(s) = \Pi(s) \zeta(s) + \beta(s) \mathbf{q}^{eq}(s) \) for some \( \zeta(s) \in \mathbb{R}^3 \) and \( \beta(s) \in \mathbb{R} \), then \( \delta^2 J[\delta \mathbf{q}] = \delta^2 J[\Pi \zeta] \). Thus, \( \mathcal{I} \) is the maximal dimension of a subspace of functions \( \zeta(s) \in \mathbb{R}^3 \) obeying \( \delta^2 J[\Pi \zeta] < 0 \), \( \zeta(0) = \zeta(1) = 0 \), and \( \langle (d_3 s')^{eq}_q, \Pi \zeta \rangle = 0 \) for \( i = 1, 2, 3 \). For convenience, we observe that the constraints \( \langle (d_3 s')^{eq}_q, \Pi \zeta \rangle = 0 \) may be rewritten as \( \langle \mathbf{T}_i, \zeta \rangle = 0 \) if we define \( \mathbf{T}_i \equiv \Pi^T (d_3 s')^{eq}_q \).

By direct computation, we find:

\[
\mathbf{T}_1(s) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{T}_2(s) = \begin{bmatrix} -2 \cos(2n \pi s) \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{T}_3(s) = \begin{bmatrix} 0 \\ -2 \sin(2n \pi s) \\ 0 \end{bmatrix}.
\]

### 3.2 The operator form of the second variation

Inserting \( \delta \mathbf{q} = \Pi \zeta \) into Equation (6), and integrating by parts terms starting with \( (\zeta')^T \), we find:

\[
\delta^2 J[\Pi \zeta] = \langle \zeta, S \zeta \rangle,
\]
where $S$ is the Sturm-Liouville operator

$$S \zeta = P \zeta'' + C \zeta' + Q \zeta,$$

with coefficient matrices:

$$P = -\Pi^T L_{q q}^{eq} \Pi,$$
$$C = (\Pi')^T L_{q q}^{eq} \Pi + \Pi^T L_{q q}^{eq} \Pi - (\Pi^T L_{q q}^{eq} \Pi + \Pi^T L_{q q}^{eq} \Pi) - (\Pi^T L_{q q}^{eq} \Pi)',$$
$$Q = (\Pi')^T L_{q q}^{eq} \Pi' + \Pi^T L_{q q}^{eq} \Pi + \Pi^T L_{q q}^{eq} \Pi' + (\Pi')^T L_{q q}^{eq} \Pi - (\Pi^T L_{q q}^{eq} \Pi + \Pi^T L_{q q}^{eq} \Pi)'.$$

Our choice of $\Pi$ causes the last term in each of $C$ and $Q$ to vanish, and yields simple $s$-independent expressions for $P, C, Q$:

$$P = \begin{pmatrix} -4K_2 & 0 & 0 \\ 0 & -4K_1 & 0 \\ 0 & 0 & -4K_3 \end{pmatrix},$$
$$C = \begin{pmatrix} 8n\pi(K_3 + K_2 - K_1) + 4K_1 u \\ 8n\pi(-2K_1 n\pi + 2K_3 n\pi + K_1 u) \\ 0 \end{pmatrix},$$
$$Q = \begin{pmatrix} -8n\pi(K_3 + K_2 - K_1) - 4K_1 u \\ 0 \end{pmatrix}.$$

Since $u$ appears linearly in $S$, it will be convenient to decompose $S$ as follows:

$$S = S_1 + u S_2,$$

where

$$S_1 \zeta = \begin{pmatrix} -4K_2 & 0 & 0 \\ 0 & -4K_1 & 0 \\ 0 & 0 & -4K_3 \end{pmatrix} \zeta'' + \begin{pmatrix} 8n\pi(K_3 + K_2 - K_1) \\ 0 \end{pmatrix} \zeta' + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \zeta,$$

and

$$S_2 \zeta = \begin{pmatrix} 0 & 0 & 4K_1 \\ 0 & 0 & 0 \\ -4K_1 & 0 & 0 \end{pmatrix} \zeta' + \begin{pmatrix} 8n\pi K_1 \\ 0 \end{pmatrix} \zeta.$$

4 Computing the index in a constrained setting

Let $\mathcal{H}$ denote $L^2((0,1), \mathbb{R}^3)$, the Hilbert space of $\mathbb{R}^3$-valued square-integrable functions. Consider the dense subspace $\mathcal{D}$ of $\mathcal{H}$ consisting of all functions $\zeta : (0,1) \to \mathbb{R}^3$ that have square-integrable
weak second derivatives and for which \( \zeta(0) = \zeta(1) = 0 \) (in Sobolev space notation, we would write \( \mathcal{D} = H^2((0,1), \mathbb{R}^3) \cap H^1_0((0,1), \mathbb{R}^3) \)). Given any subspace \( V \) of \( \mathcal{D} \), let \( V^\perp \) denote the orthogonal complement of \( V \) in \( \mathcal{D} \).

The second variation operator \( \mathcal{S} \) is a map from \( \mathcal{D} \) into \( \mathcal{H} \). Let \( R(\mathcal{S}) \) and \( ker(\mathcal{S}) \) denote the range and kernel of \( \mathcal{S} \), respectively. Let \( \mathcal{T} = \text{span}(T_1, T_2, T_3) \). The constrained index \( \mathcal{I} \) we wish to compute is the maximal dimension of a subspace of \( \mathcal{T}^\perp \) on which \( \langle \zeta, \mathcal{S}\zeta \rangle < 0 \).

Maddock's [8] provides a formula for \( \mathcal{I} \) in this setting, assuming several hypotheses on \( \mathcal{S} \):

(H1) \( \mathcal{S} \) is self-adjoint and satisfies Fredholm’s alternative \( (R(\mathcal{S}) = ker(\mathcal{S})^\perp) \),

(H2) \( \mathcal{S} \) has a finite number \( \mathcal{I}_u \) of orthonormal eigenvectors \( \zeta_i^- \) corresponding to negative eigenvalues,

(H3) \( \langle \zeta, \mathcal{S}\zeta \rangle > 0 \) on \( (\text{span}\{\zeta_i^-\} \oplus ker(\mathcal{S}))^\perp \).

Note that \( \mathcal{I}_u \) is the maximal dimension of a subspace of \( \mathcal{D} \) on which \( \langle \zeta, \mathcal{S}\zeta \rangle < 0 \); thus it can be thought of as an unconstrained index, as it is analogous to the index \( \mathcal{I} \), but disregarding the isoperimetric constraints \( \langle T_i, \zeta \rangle = 0 \).

In our example, the self-adjointness of \( \mathcal{S} \) is evident from the expressions for \( \mathbf{P} \), \( \mathbf{C} \), and \( \mathbf{Q} \) (since \( \mathbf{P} \) and \( \mathbf{Q} \) are symmetric and \( \mathbf{C} \) is skew). To show that \( \mathcal{S} \) satisfies the Fredholm alternative, we appeal to a regularity estimate \( \|u\|_{H^2 \cap H^1_0} \leq C(\|Su\|_{L^2} + \|u\|_{L^2}) \) (see, e.g., [7] for the vector version of this standard scalar result), from which it follows that \( R(\mathcal{S}) \) is closed (see, for example, [6, p. 231] or [10]). Since \( R(\mathcal{S}) \) is closed, the closed range theorem [6, p. 234] guarantees that \( \mathcal{S} \) satisfies the Fredholm alternative. Hence, (H1) is satisfied. The hypothesis (H2) follows from the fact that the spectrum of a Sturm-Liouville operator consists of a countable set of eigenvalues \( \lambda_n \) that are bounded below and accumulate only at \( +\infty \) (see, e.g., [3] for the vector version of this standard scalar result). The third hypothesis (H3) is a direct consequence of the spectral theorem for self-adjoint operators [6, p. 360] since \( (\text{span}\{\zeta_i^-\} \oplus ker(\mathcal{S}))^\perp \) is spanned by the eigenvectors associated with the positive eigenvalues.

Maddock’s formula is [8, Theorem 2, Lemma 6]

\[
\mathcal{I} = \mathcal{I}_u - d^{\text{wp}}(W),
\]

Here, \( d^{\text{wp}}(W) \) is the number of nonpositive eigenvalues of the 3-by-3 matrix

\[
W \equiv \begin{bmatrix} \langle \eta_1, T_1 \rangle & \langle \eta_1, T_2 \rangle & \langle \eta_1, T_3 \rangle \\ \langle \eta_2, T_1 \rangle & \langle \eta_2, T_2 \rangle & \langle \eta_2, T_3 \rangle \\ \langle \eta_3, T_1 \rangle & \langle \eta_3, T_2 \rangle & \langle \eta_3, T_3 \rangle \end{bmatrix},
\]

where \( \eta_i \) is a solution of the inhomogeneous problem \( \mathcal{S}\eta_i = T_j, \eta_i(0) = \eta_i(1) = 0 \).
5 General solution to the homogeneous problem

The determination of both $I_u$ and $d^{uv}(W)$ will use the solution of the homogeneous equation $S\zeta = 0$, which, written out explicitly (dividing through by $4K_1$), is:

\[-\rho \zeta_1'' + [2n\pi(\gamma + \rho - 1) + \hat{u}]\zeta_1' + 2n\pi[2n\pi(\gamma - 1) + \hat{u}]\zeta_1 = 0,
\]
\[-\gamma \zeta_2'' - [2n\pi(\gamma + \rho - 1) + \hat{u}]\zeta_2' + 2n\pi[2n\pi(\rho - 1) + \hat{u}]\zeta_3 = 0,
\]

where $\rho = K_2/K_1$ and $\gamma = K_3/K_1$. The second equation implies $\zeta_2(s) = \alpha s + \beta$ for some $\alpha, \beta$.

Define $\eta_1 = \zeta_1'$ and $\eta_3 = \zeta_3'$, to rewrite the second-order equations for $\zeta_1, \zeta_3$ as a first-order system:

\[
\frac{d}{ds} \begin{bmatrix} \zeta_1 \\ \zeta_3 \\ \eta_1 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4n^2\pi^2(1 + \hat{w}) & 0 & 0 & 2n\pi(\delta + 1 + \hat{w}) \\ 0 & 4n^2\pi^2(\frac{1}{\delta} + \hat{w}) & -2n\pi(1 + \frac{1}{\delta} + \hat{w}) & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_3 \\ \eta_1 \\ \eta_3 \end{bmatrix}, \tag{11}
\]

where

\[\hat{w} \equiv \frac{\hat{u} - 2n\pi}{2n\pi\gamma}, \quad \delta \equiv \frac{\gamma}{\rho} .\]

The form of the solution to Eq. (11) depends on the value of

\[D \equiv (\hat{w} + 1)(\delta\hat{w} + 1) .\]

**Case 1: $D < 0$**

The matrix in (11) has eigenvalues $(\pm 2n\pi i, \pm 2n\pi \sqrt{-D})$ and the general solution is:

\[
\zeta_1 = C_1 \cos(2n\pi s) + C_2 \sin(2n\pi s) - C_3 \sqrt{-D} e^{2n\pi s \sqrt{-D}} - C_4 \sqrt{-D} e^{-2n\pi s \sqrt{-D}},
\]
\[
\zeta_3 = -C_1 \sin(2n\pi s) + C_2 \cos(2n\pi s) + C_3 (1 + \hat{w}) e^{2n\pi s \sqrt{-D}} - C_4 (1 + \hat{w}) e^{-2n\pi s \sqrt{-D}} . \tag{12}
\]

**Case 2a: $D = 0, \hat{w} = -\frac{1}{\delta} (\hat{u} = 2n\pi(1 - \rho))$**

The matrix in (11) has eigenvalues $(\pm 2n\pi i, 0, 0)$. If $\delta \neq 1$, the general solution is:

\[
\zeta_1 = C_1 \cos(2n\pi s) + C_2 \sin(2n\pi s) + C_4 \frac{\delta}{1 - \delta}, \tag{13}
\]
\[
\zeta_3 = -C_1 \sin(2n\pi s) + C_2 \cos(2n\pi s) + C_3 + 2n\pi C_4 s,
\]

while if $\delta = 1$, the general solution is:

\[
\zeta_1 = C_1 \cos(2n\pi s) + C_2 \sin(2n\pi s) + C_4,
\]
\[
\zeta_3 = -C_1 \sin(2n\pi s) + C_2 \cos(2n\pi s) + C_3 . \tag{14}
\]

**Case 2b: $D = 0, \hat{w} = -1 (\hat{u} = 2n\pi(1 - \gamma))$**

The matrix in (11) has eigenvalues $(\pm 2n\pi i, 0, 0)$. If $\delta \neq 1$, the general solution is:

\[
\zeta_1 = C_1 \cos(2n\pi s) + C_2 \sin(2n\pi s) + C_3 + 2n\pi C_4 s,
\]
\[
\zeta_3 = -C_1 \sin(2n\pi s) + C_2 \cos(2n\pi s) + C_4 \frac{1}{1 - \delta} . \tag{15}
\]
while if $\delta = 1$ ($\gamma = \rho$), the general solution is given by Eq. (14) from Case 2a.

Case 3: $D > 0, D \neq 1$

The matrix in (11) has eigenvalues ($\pm 2n \pi i, \pm 2n \pi \sqrt{D} i$), and the general solution is:

$$\zeta_1 = C_1 \cos(2n \pi s) + C_2 \sin(2n \pi s) - C_3 \sqrt{D} \cos(2n \pi s \sqrt{D}) - C_4 \sqrt{D} \sin(2n \pi s \sqrt{D})$$
$$\zeta_3 = -C_1 \sin(2n \pi s) + C_2 \cos(2n \pi s) + C_3 (1 + \hat{w}) \sin(2n \pi s \sqrt{D}) - C_4 (1 + \hat{w}) \cos(2n \pi s \sqrt{D}).$$  

(16)

Case 4a: $D = 1, \hat{w} = 0$ ($\hat{u} = 2n \pi$)

The matrix in (11) has eigenvalues ($\pm 2n \pi i, \pm 2n \pi i$), and the general solution is:

$$\zeta_1 = C_1 \cos(2n \pi s) + C_2 \sin(2n \pi s) + C_3 \left( \frac{-\sin(2n \pi s)}{n \pi (\delta + 1)} + s \cos(2n \pi s) \right)$$
$$+ C_4 \left( \frac{-\cos(2n \pi s)}{n \pi (1 + \delta)} + s \sin(2n \pi s) \right)$$

$$\zeta_3 = -C_1 \sin(2n \pi s) + C_2 \cos(2n \pi s) + C_3 \left( -\frac{-\cos(2n \pi s)}{2n \pi} - s \sin(2n \pi s) \right)$$
$$+ C_4 \left( \frac{-\sin(2n \pi s)}{2n \pi} + s \cos(2n \pi s) \right).$$  

(17)

Case 4b: $D = 1, \hat{w} = -(1 + \frac{1}{\delta})$ ($\hat{u} = 2n \pi (1 - \rho - \gamma)$)

The matrix in (11) has eigenvalues ($\pm 2n \pi i, \pm 2n \pi i$), and the general solution is:

$$\zeta_1 = C_1 \cos(2n \pi s) + C_2 \sin(2n \pi s) + C_3 \cos(2n \pi s) + C_4 \sin(2n \pi s)$$
$$\zeta_3 = -C_1 \sin(2n \pi s) + C_2 \cos(2n \pi s) + C_3 \sin(2n \pi s) - C_4 \cos(2n \pi s).$$  

(18)

6 Unconstrained index

In this section, we compute $I_u$, the maximal dimension of a subspace of $D$ on which $\langle \zeta, \mathcal{S}\zeta \rangle < 0$. This unconstrained index is computed by determining a convenient basis for $D$. The key members of this basis are the eigenvectors of the eigenvalue problem

$$\mathcal{S}\zeta = S_1 \zeta + \hat{w}\mathcal{S}\zeta = 0, \ \zeta(0) = \zeta(1) = 0.$$  

(19)

Eq. (19) is reminiscent of a Sturm-Liouville problem, although in that case, $S_2 \zeta$ would equal $A \zeta$ for some positive definite matrix $A$. We will show that for a given $\hat{u}$, $I_u$ is the number of eigenvectors for which $\langle \zeta, \mathcal{S}\zeta \rangle < 0$.

6.1 Solutions to the eigenvalue problem (19)

The general solution of $\mathcal{S}\zeta = 0$ has $\zeta_2(s) = \alpha s + \beta$, and we see immediately that $\zeta(0) = \zeta(1) = 0$ implies that $\alpha = \beta = 0$. Therefore, we restrict attention to $\zeta_1, \zeta_3$. Using the general solutions
(12–18), we write the boundary conditions in the form:
\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
\zeta_1(0) \\
\zeta_3(0) \\
\zeta_1(1) \\
\zeta_3(1)
\end{bmatrix}
= \mathbf{M}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix}.
\]
This has a nontrivial solution if and only if the determinant of the 4-by-4 matrix \(\mathbf{M}\) is zero.

**Case 1:** \(D < 0\)

\[
\det \mathbf{M} = \det
\begin{bmatrix}
1 & 0 & -\sqrt{-D} & -\sqrt{-D} \\
0 & 1 & 1 + \hat{\omega} & -(1 + \hat{\omega}) \\
1 & 0 & -\sqrt{-D} e^{2\pi\sqrt{-D}} & -\sqrt{-D} e^{-2\pi\sqrt{-D}} \\
0 & 1 & (1 + \hat{\omega}) e^{2\pi\sqrt{-D}} & -(1 + \hat{\omega}) e^{-2\pi\sqrt{-D}}
\end{bmatrix}
= -2(1 + \hat{\omega}) \sqrt{-D} [e^{\pi\sqrt{-D}} - e^{-\pi\sqrt{-D}}]^2 \neq 0,
\]
so there are no solutions to (19) in this case.

**Case 2a:** \(D = 0, \hat{\omega} = -\frac{1}{\delta} (\hat{u} = 2n\pi(1 - \rho))\)

For \(\delta = 1\),

\[
\det \mathbf{M} = \det
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix} = 0,
\]
and \(\mathbf{M}\) has a two-dimensional nullspace spanned by \((C_1, C_2, C_3, C_4) = (1, 0, 0, -1)\) and \((0, 1, -1, 0)\). Thus, \(\hat{u} = 2n\pi(1 - \rho)\) is an eigenvalue for (19) with corresponding eigenvectors

\[
\begin{bmatrix}
\zeta_1 \\
\zeta_3
\end{bmatrix}
= \begin{bmatrix}
\cos(2n\pi s) - 1 \\
\sin(2n\pi s)
\end{bmatrix}, \begin{bmatrix}
\sin(2n\pi s) \\
\cos(2n\pi s) - 1
\end{bmatrix}.
\]

For \(\delta \neq 1\),

\[
\det \mathbf{M} = \det
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{\delta} \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & \frac{1}{\delta} \\
0 & 1 & 1 & 2n\pi
\end{bmatrix} = 0,
\]
and \(\mathbf{M}\) has a one-dimensional nullspace spanned by \((C_1, C_2, C_3, C_4) = (0, 1, -1, 0)\). Hence, \(\hat{u} = 2n\pi(1 - \rho)\) is an eigenvalue for (19) with corresponding eigenvector

\[
\begin{bmatrix}
\zeta_1 \\
\zeta_3
\end{bmatrix}
= \begin{bmatrix}
\sin(2n\pi s) \\
\cos(2n\pi s) - 1
\end{bmatrix}.
\]

**Case 2b:** \(D = 0, \hat{\omega} = -1 (\hat{u} = 2n\pi(1 - \gamma))\)

For \(\delta = 1\), Case 2b reduces to Case 2a. For \(\delta \neq 1\),

\[
\det \mathbf{M} = \det
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & \frac{1}{\delta} \\
1 & 0 & 1 & 2n\pi \\
0 & 1 & 0 & \frac{1}{1 - \delta}
\end{bmatrix} = 0,
\]
and \( M \) has a one-dimensional nullspace spanned by \((C_1, C_2, C_3, C_4) = (1, 0, -1, 0)\). Hence, \( \hat{u} = 2n\pi(1 - \gamma) \) is an eigenvalue for (19) with corresponding eigenvector:

\[
\begin{bmatrix}
\zeta_1 \\
\zeta_3
\end{bmatrix} = \begin{bmatrix}
\cos(2n\pi s) - 1 \\
-\sin(2n\pi s)
\end{bmatrix}.
\]

**Case 3:** \( D > 0, D \neq 1 \)

\[
\det M = \det \begin{bmatrix}
1 & 0 & -\sqrt{D} & 0 \\
0 & 1 & 0 & -(1 + \hat{w}) \\
1 & 0 & -\sqrt{D} \cos(2n\pi \sqrt{D}) & -\sqrt{D} \sin(2n\pi \sqrt{D}) \\
0 & 1 & (1 + \hat{w}) \sin(2n\pi \sqrt{D}) & -(1 + \hat{w}) \cos(2n\pi \sqrt{D})
\end{bmatrix} = 4\sqrt{D}(1 + \hat{w}) \sin^2(n\pi \sqrt{D}),
\]

which equals zero when \( \sqrt{D} = \frac{k}{n} \) for some positive integer \( k \neq n \) \((k = n \text{ is excluded because } D \neq 1 \text{ in Case 3})\). Solving for \( \hat{w} \) (or equivalently \( \hat{u} \)) in the equation \( \sqrt{D} = \frac{k}{n} \), we find the following eigenvalues for (19):

\[
\hat{w} = \hat{w}^{n,k,\pm} \equiv \frac{(1 + \delta) \pm \sqrt{(1 - \delta)^2 + 4\delta k^2}}{2\delta}, \tag{20}
\]

or

\[
\hat{u} = \hat{u}^{n,k,\pm} \equiv (2 - \gamma - \rho)n\pi \pm \pi \sqrt{n^2(\gamma - \rho)^2 + 4\gamma \rho k^2}.
\]

At these eigenvalues, the matrix \( M \) has a two-dimensional nullspace spanned by \((C_1, C_2, C_3, C_4) = (1, 0, \frac{2}{k}, 0)\) and \((0, 1, 0, \frac{1}{1 + \hat{w}})\), and thus, the corresponding eigenvectors are:

\[
\begin{bmatrix}
\zeta_1 \\
\zeta_3
\end{bmatrix} = \begin{bmatrix}
\cos(2n\pi s) - \cos(2k\pi s) \\
-\sin(2n\pi s) + \frac{n(1 + \hat{w})}{k} \sin(2k\pi s)
\end{bmatrix}, \quad \begin{bmatrix}
\sin(2n\pi s) - \frac{k}{n(1 + \hat{w})} \sin(2k\pi s) \\
\cos(2n\pi s) - \cos(2k\pi s)
\end{bmatrix}.
\]

**Case 4a:** \( D = 1, \hat{w} = 0 \) (\( \hat{u} = 2n\pi \))

\[
\det M = \det \begin{bmatrix}
1 & 0 & 0 & \frac{1}{1 + \delta} \\
0 & 1 & -\frac{1}{2n\pi} & 0 \\
1 & 0 & 1 & \frac{1}{1 + \delta} \\
0 & 1 & 1 & 1
\end{bmatrix} = 1 \neq 0,
\]

so there are no solutions to (19) in this case.

**Case 4b:** \( D = 1, \hat{w} = -(1 + \frac{1}{k}) \) (\( \hat{u} = 2n\pi(1 - \rho - \gamma) \))

\[
\det M = \det \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix} = 0,
\]

11
and \( \mathbf{M} \) has a two-dimensional nullspace spanned by \((C_1, C_2, C_3, C_4) = (1, 0, -1, 0) \) and \((0, 1, 0, 1) \). Hence, \( \hat{u} = 2n\pi(1 + \rho - \gamma) \) is an eigenvalue of (19) with corresponding eigenvectors:

\[
\begin{bmatrix}
\zeta_1 \\
\zeta_3
\end{bmatrix} = \begin{bmatrix}
0 \\
\sin(2n\pi s)
\end{bmatrix}, \quad \begin{bmatrix}
\sin(2n\pi s) \\
0
\end{bmatrix}.
\]

In summary, combining Cases 2a, 2b, 3, and 4b, the eigenvalues are \( \hat{u} = \hat{u}^{n,k}+ \) for \( k \geq 0, k \neq n \), and \( \hat{u} = \hat{u}^{n,k,-} \) for \( k \geq 0 \) (note that if \( \delta = 1 \), then \( \hat{u}^{n,0,-} = \hat{u}^{n,0,+} \)). These eigenvalues are ordered as follows:

\[
... \hat{u}^{n,2,-} < \hat{u}^{n,1,-} < \hat{u}^{n,0,-} \leq \hat{u}^{n,0,+} < \hat{u}^{n,1,+} < \hat{u}^{n,2,+} < \cdots < \hat{u}^{n,n-1,+} < \hat{u}^{n,n,+} < \hat{u}^{n,n+1,+} < \hat{u}^{n,n+2,+} < ... \]

For \( k > 0 \), there are two eigenvectors for each eigenvalue:

\[
\zeta^{n,k_{;\pm,1}}(s) = \begin{bmatrix}
\cos(2n\pi s) - \cos(2k\pi s) \\
0 \\
-\sin(2n\pi s) + \frac{n(1+\theta^{n,k;\pm})}{k} \sin(2k\pi s)
\end{bmatrix}, \quad \zeta^{n,k_{;\pm,2}}(s) = \begin{bmatrix}
\sin(2n\pi s) - \frac{k}{n(1+\theta^{n,k;\pm})} \sin(2k\pi s) \\
0 \\
\cos(2n\pi s) - \cos(2k\pi s)
\end{bmatrix}.
\]

For \( k = 0 \), there is a single eigenvector for each eigenvalue. For \( \delta > 1 \),

\[
\zeta^{n,0,+}(s) = \begin{bmatrix}
\sin(2n\pi s) \\
0 \\
\cos(2n\pi s) - 1
\end{bmatrix}, \quad \zeta^{n,0,-}(s) = \begin{bmatrix}
\cos(2n\pi s) - 1 \\
0 \\
-\sin(2n\pi s)
\end{bmatrix},
\]

while for \( \delta < 1 \) they are swapped, and for \( \delta = 1 \), the eigenvalues \( \hat{u}^{n,0,\pm} \) merge, and we have a two-dimensional eigenspace spanned by these two vectors.

### 6.2 The sign of the second variation on the eigenvectors

For each eigenvector \( \zeta \), we determine the sign of \( \langle \zeta, S_{2}\zeta \rangle \). We relate this to the sign of \( \langle \zeta, S_{2}\zeta \rangle \) by exploiting the fact that \((S_{1} + \hat{u}^{n,k;\pm} S_{2}) \zeta^{n,k_{;\pm,j}} = 0:

\[
\langle \zeta^{n,k_{;\pm,j}}, S_{2} \zeta^{n,k_{;\pm,j}} \rangle = \langle \zeta^{n,k_{;\pm,j}}, (S_{1} + \hat{u} S_{2}) \zeta^{n,k_{;\pm,j}} \rangle = \langle \zeta^{n,k_{;\pm,j}}, (S_{1} + \hat{u}^{n,k;\pm} S_{2} - \hat{u}^{n,k;\pm} S_{2} + \hat{u} S_{2}) \zeta^{n,k_{;\pm,j}} \rangle = (\hat{u} - \hat{u}^{n,k;\pm}) \langle \zeta^{n,k_{;\pm,j}}, S_{2} \zeta^{n,k_{;\pm,j}} \rangle.
\]

#### Case I: \( \zeta^{n,k_{;\pm,1}} \) for \( k > 0 \)

We compute

\[
S_{2} \zeta^{n,k_{;\pm,1}} = \begin{bmatrix}
8n\pi K_{1} \hat{u}^{n,k;\pm} \cos(2k\pi s) \\
0 \\
\frac{8n\pi K_{1} \sin(2k\pi s)}{k} (n^{2}(1 + \hat{u}^{n,k;\pm}) - k^{2})
\end{bmatrix},
\]

and therefore:

\[
\langle \zeta^{n,k_{;\pm,1}}, S_{2} \zeta^{n,k_{;\pm,1}} \rangle = 4K_{1} n\pi \left( \frac{n^{2}}{k^{2}} (1 + \hat{u}^{n,k;\pm})^{2} - 2(1 + \hat{u}^{n,k;\pm}) + 1 \right)
\]

(22)
Lemma 1. \( \frac{n^2}{k^2} (1 + \hat{w}^{n,k,-})^2 - 2(1 + \hat{w}^{n,k,-}) + 1 \) is positive for \( k = 1, 2, \ldots \).

Proof:
This follows immediately from the fact that \( 1 + \hat{w}^{n,k,-} < 0 \). \( \square \)

Lemma 2. \( \frac{n^2}{k^2} (1 + \hat{w}^{n,k,+})^2 - 2(1 + \hat{w}^{n,k,+}) + 1 \) is positive for \( k = 1, 2, \ldots, n-1 \) and is negative for \( k = n+1, n+2, \ldots \).

Proof:
If \( k \leq n \), then
\[
\frac{n^2}{k^2} (1 + \hat{w}^{n,k,+})^2 - 2(1 + \hat{w}^{n,k,+}) + 1 > (1 + \hat{w}^{n,k,+})^2 - 2(1 + \hat{w}^{n,k,+}) + 1 = (\hat{w}^{n,k,+})^2 > 0.
\]

Now assume \( k > n \). Using (20) we compute:
\[
\frac{n^2}{k^2} (1 + \hat{w}^{n,k,+})^2 - 2(1 + \hat{w}^{n,k,+}) + 1 = \frac{n^2}{k^2} \left[(1 - \delta)^2 + \frac{4\delta k^2}{n^2} \right] - (\frac{n^2}{k^2} (1 - \delta) + 2\delta) \sqrt{(1 - \delta)^2 + \frac{4\delta k^2}{n^2}}
\]
\[
= \frac{n^2}{2\delta^2} \left( \sqrt{(1 - \delta)^2 + \frac{4\delta k^2}{n^2}} - \left[ (1 - \delta) + \frac{2\delta k^2}{n^2} \right] \right)
\]
This expression is negative, since:
\[
k > n \implies \frac{4\delta^2 k^2}{n^2} < \frac{4\delta^2 k^4}{n^4}
\]
\[
\implies (1 - \delta)^2 + \frac{4\delta k^2}{n^2} < (1 - \delta)^2 + \frac{4\delta k^2}{n^2} + \left( \frac{4\delta^2 k^4}{n^4} - \frac{4\delta^2 k^2}{n^2} \right) = (1 - \delta)^2 + \frac{4\delta k^2}{n^2} (1 - \delta) + \frac{4\delta^2 k^4}{n^4}
\]
\[
\implies \sqrt{(1 - \delta)^2 + \frac{4\delta k^2}{n^2}} < (1 - \delta) + \frac{2\delta k^2}{n^2}.
\]
\( \square \)

By Lemma 1 and Eq. (21), \( \langle \xi^{n,k,-1}, S \xi^{n,k,-1} \rangle \) has the same sign as \( \hat{u} - \hat{w}^{n,k,-} \) for all \( k > 0 \), and by Lemma 2, \( \langle \xi^{n,k,+1}, S \xi^{n,k,+1} \rangle \) has the same sign as \( \hat{u} - \hat{w}^{n,k,+} \) for \( 0 < k < n \), and the opposite sign as \( \hat{u} - \hat{w}^{n,k,+} \) for \( k > n \).

Case II: \( \xi^{n,k,\pm 2} \) for \( k > 0 \)

We find that
\[
S_{\xi^{n,k,\pm 2}} = \begin{bmatrix}
\frac{8\pi K_1 \hat{w}^{n,k,\pm}}{1 + \hat{w}^{n,k,\pm}} \sin(2k\pi s) & 0 \\
\frac{8\pi K_1 \cos(2k\pi s)}{n(1 + \hat{w}^{n,k,\pm})} & \frac{\hat{w}^{n,k,\pm}}{n(1 + \hat{w}^{n,k,\pm})} (n^2 (1 + \hat{w}^{n,k,\pm}) + k^2)
\end{bmatrix},
\]
and therefore
\[
\langle \xi^{n,k,\pm 2}, S_{\xi^{n,k,\pm 2}} \rangle = \frac{4k^2 \pi K_1}{n(1 + \hat{w}^{n,k,\pm})^2} \left( \frac{n^2}{k^2} (1 + \hat{w}^{n,k,\pm})^2 - 2(1 + \hat{w}^{n,k,\pm}) + 1 \right) \quad (23)
\]
So, by Lemma 1 and Eq. (21), \( \langle \zeta^n_{k;r}, \zeta^n_{k;r-2} \rangle \) has the same sign as \( \hat{u} - \hat{u}^n_{k;r} \) for all \( k > 0 \), and by Lemma 2 \( \langle \zeta^n_{k;r+1^2}, \zeta^n_{k;r+2^2} \rangle \) has the same sign as \( \hat{u} - \hat{u}^n_{k;r} \) for \( 0 < k < n \), and the opposite sign as \( \hat{u} - \hat{u}^n_{k;r+1} \) for \( k > n \).

**Case III:** \( \zeta^n_{0\pm} \)

Depending on whether or not \( \delta > 1 \), one of these equals \( (\sin(2n\pi s), 0, \cos(2n\pi s) - 1) \) and the other equals \( (\cos(2n\pi s) - 1, 0, -\sin(2n\pi s)) \). The first vector gives: \( S_2 \zeta = (0, 0, -8n\pi K_1) \), while the second vector gives: \( S_2 \zeta = (-8n\pi K_1, 0, 0) \), and in both cases,

\[
\langle \zeta^n_{0\pm}, S_2 \zeta^n_{0\pm} \rangle = 8n\pi K_1.
\]

So, in both cases, by Eq. (21), \( \langle \zeta^n_{0\pm}, S_2 \zeta^n_{0\pm} \rangle \) has the same sign as \( \hat{u} - \hat{u}^n_{0\pm} \).

In summary of Cases I–III, \( \langle \zeta^n_{k;r\pm}, \zeta^n_{k;r\pm} \rangle \) has the same sign as \( \hat{u} - \hat{u}^n_{k;r\pm} \) in all cases except for \( \langle \zeta^n_{k;r+1^2}, \zeta^n_{k;r+2^2} \rangle \) with \( k > n \). Based on this result, we list in Table 1 those eigenvectors giving a negative second variation \( \langle \zeta, S_2 \zeta \rangle \).

<table>
<thead>
<tr>
<th>Range of ( \hat{u} )</th>
<th>Eigenvectors for which ( \langle \zeta, S_2 \zeta \rangle &lt; 0 )</th>
<th>Number of eigenvectors for which ( \langle \zeta, S_2 \zeta \rangle &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{u}^{n,m+1^2} \leq \hat{u} \leq \hat{u}^{n,m+1^2} ) ( (m \geq n + 1) )</td>
<td>( \zeta^{n,m+1^2}, \ldots, \zeta^{n,m+1^2} ) ( (j = 1, 2) )</td>
<td>( 2(m - n) )</td>
</tr>
<tr>
<td>( \hat{u}^{n,m-1^2} \leq \hat{u} \leq \hat{u}^{n,m+1^2} ) ( (0 \leq m \leq n - 2) )</td>
<td>( \zeta^{n,m+1^2}, \ldots, \zeta^{n,m+1^2} ) ( (j = 1, 2) )</td>
<td>( 2(n - m - 1) )</td>
</tr>
<tr>
<td>( \hat{u}^{n,0^2} \leq \hat{u} \leq \hat{u}^{n,0^2} )</td>
<td>( \zeta^{n,1^2}, \ldots, \zeta^{n,1^2} ) ( (j = 1, 2) ) ( \text{and} ) ( \zeta^{n,0^2} )</td>
<td>( 2n - 1 )</td>
</tr>
<tr>
<td>( \hat{u}^{n,m+1^2} \leq \hat{u} \leq \hat{u}^{n,m+1^2} ) ( (m \geq 0) )</td>
<td>( \zeta^{n,m+1^2}, \ldots, \zeta^{n,m+1^2} ) ( (j = 1, 2) ) ( \text{and} ) ( \zeta^{n,0^2}, \ldots, \zeta^{n,0^2} ) ( (j = 1, 2) )</td>
<td>( 2n + 2m )</td>
</tr>
</tbody>
</table>

**Table 1**: Eigenvectors \( \zeta \) giving a negative second variation

### 6.3 \( S_2 \)-orthogonality of the eigenvectors

In the remainder of Sec. 6, we show that the number of eigenvectors on which the second variation is negative is equal to the unconstrained index. As a first step, we show that these eigenvectors are \( S_2 \)-orthogonal.

For a pair of eigenvectors \( \zeta, \zeta^* \) arising from distinct eigenvalues \( \hat{u}, \hat{u}^* \) of the eigenvalue problem (19), a standard argument demonstrates \( S_2 \)-orthogonality:

\[
\hat{u}^* \langle \zeta^*, S_2 \zeta \rangle = \hat{u}^* \langle S_2 \zeta^*, \zeta \rangle = -\langle S_1 \zeta^*, \zeta \rangle = -\langle \zeta^*, S_1 \zeta \rangle = \hat{u} \langle \zeta^*, S_2 \zeta \rangle
\]

(25)

(using self-adjointness of \( S_1 \) and \( S_2 \)), and thus \( \langle \zeta^*, S_2 \zeta \rangle = 0 \) since \( \hat{u} \neq \hat{u}^* \).
For \( k > 0, \ k \neq n \), the eigenvectors \( \zeta^{n,k+1}, \zeta^{n,k+2} \) share the common eigenvalue \( \hat{\omega}^{n,k+} \), so the above argument does not hold, but we may verify \( S_2 \)-orthogonality directly:

\[
\langle \zeta^{n,k+2}, S_2 \zeta^{n,k+1,1} \rangle = \int_0^1 \begin{bmatrix} \sin(2n\pi s) - \frac{k}{n(1 + \hat{\omega}^{n,k+})} \sin(2k\pi s) \\ \cos(2n\pi s) - \cos(2k\pi s) \end{bmatrix} \cdot \begin{bmatrix} 8\pi nK_1 \hat{\omega}^{n,k+} \cos(2k\pi s) \\ 0 \end{bmatrix} ds
= 0
\]

Similarly, for \( k > 0 \), the eigenvectors \( \zeta^{n,k,-1}, \zeta^{n,k,-2} \) share the common eigenvalue \( \hat{\omega}^{n,k-} \), and we again verify \( S_2 \)-orthogonality directly (the computation is identical to the one shown above, with \( \hat{\omega}^{n,k+} \) replaced by \( \hat{\omega}^{n,k-} \)).

### 6.4 Basis construction from eigenvectors

In analogy with Sturm-Liouville problems, we use the eigenvectors to form a basis for \( D \). Since the eigenvectors vanish in the second slot, we adjoin to them a Fourier basis in the second slot:

\[
C \equiv \left\{ \begin{bmatrix} 0 \\ 1 - \cos(2k\pi s) \end{bmatrix}, \begin{bmatrix} 0 \\ \sin(2k\pi s) \end{bmatrix} : k \in \mathbb{N} \right\}.
\]

We will show that the collection \( G = \{ \zeta^{n,k\pm,j} \} \cup C \) is an \( S \)-orthogonal basis for \( D \).

The mutual \( S \)-orthogonality of the eigenvectors \( \{ \zeta^{n,k\pm,j} \} \) follows immediately from Sec. 6.3, since \( \langle \zeta^*, S_2 \zeta \rangle = 0 \) implies that \( \langle \zeta^*, S_1 \zeta \rangle = 0 \) by (25), and thus \( \langle \zeta^*, S \zeta \rangle = \langle \zeta^*, (S_1 + \hat{\omega}S_2) \zeta \rangle = 0 \).

In addition, mutual \( S \)-orthogonality of members of \( C \), or \( S \)-orthogonality of a member of \( C \) with an eigenvector, may be easily verified directly, using the expressions

\[
S \begin{bmatrix} 0 \\ 1 - \cos(2k\pi s) \end{bmatrix} = \begin{bmatrix} 0 \\ 16K_1k^2\pi^2 \cos(2k\pi s) \end{bmatrix}, \quad S \begin{bmatrix} 0 \\ \sin(2k\pi s) \end{bmatrix} = \begin{bmatrix} 0 \\ 16K_1k^2\pi^2 \sin(2k\pi s) \end{bmatrix}.
\]

(26)

Linear independence of any finite collection of eigenvectors can be established directly from their \( S_2 \)-orthogonality: suppose some eigenvector \( \zeta \) could be expressed as a finite linear combination of other eigenvectors; apply \( S_2 \) and take the inner product with \( \zeta \) to find that \( \langle \zeta, S_2 \zeta \rangle = 0 \), in contradiction to Eqs. (22,23,24). Since the elements of \( C \) are linearly independent by Fourier analysis, and are orthogonal to the eigenvectors, any finite collection of elements of \( G \) is linearly independent.

It may be directly verified that any member of the collection

\[
B \equiv \left\{ \begin{bmatrix} 1 - \cos(2k\pi s) \\ 0 \\ \sin(2k\pi s) \end{bmatrix}, \begin{bmatrix} 0 \\ 1 - \cos(2k\pi s) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \sin(2k\pi s) \end{bmatrix} : k \in \mathbb{N} \right\},
\]

is \( S \)-orthogonal.
can be expressed as a finite linear combination of elements in \( \mathcal{G} \). For example, for \( k \neq n \),

\[
\begin{bmatrix}
1 - \cos(2k\pi s) \\
0 \\
0
\end{bmatrix}
= \frac{1 + \hat{w}^n_k k_s - \hat{w}^n_k k_{s-1}}{\hat{w}^n_k k_s - \hat{w}^n_k k_{s-1}} \zeta^n_{k,s+1} - \frac{1 + \hat{w}^n_k k_s + \hat{w}^n_k k_{s-1}}{\hat{w}^n_k k_s + \hat{w}^n_k k_{s-1}} \zeta^n_{k,s-1} - \begin{bmatrix}
\cos(2n\pi s) - 1 \\
0 \\
-\sin(2n\pi s)
\end{bmatrix},
\]

the latter term equaling either \( \zeta^n k_s^- \) or \( \zeta^n k_s^+ \) depending on \( \delta \). Since by Fourier analysis, \( \mathcal{B} \) forms an orthonormal basis for \( \mathcal{D} \), the collection \( \mathcal{G} \) is also a basis for \( \mathcal{D} \), i.e., any function in \( \mathcal{D} \) can be approximated arbitrarily accurately (in \( L^2 \)) by a finite linear combination of functions in \( \mathcal{G} \).

### 6.5 Determination of the unconstrained index

For any \( \hat{u} \), let \( \mathcal{N} \) denote the subspace spanned by those eigenvectors for which the second variation \( \langle \zeta, S\zeta \rangle \) is negative (see Table 1).

**Lemma 3.** \( \mathcal{N} \) is a maximal subspace on which the second variation is negative.

**Proof:** Any \( \chi(s) \) orthogonal to \( \mathcal{N} \) can be expanded in the basis of the remaining elements of \( \mathcal{G} \):

\[
\chi = \sum_{\zeta \notin \mathcal{N}} c_\zeta \zeta + \sum_{\eta \in \mathcal{C}} c_\eta \eta
\]

(where the notation \( \zeta \notin \mathcal{N} \) denotes the collection of those eigenvectors that are not in \( \mathcal{N} \)). Then, by \( S \)-orthogonality of these vectors,

\[
\langle \chi, S\chi \rangle = \sum_{\zeta \notin \mathcal{N}} (c_\zeta)^2 \langle \zeta, S\zeta \rangle + \sum_{\eta \in \mathcal{C}} (c_\eta)^2 \langle \eta, S\eta \rangle.
\]

By definition of \( \mathcal{N} \), \( \langle \zeta, S\zeta \rangle \geq 0 \) for eigenvectors \( \zeta \) not in \( \mathcal{N} \). In addition, for \( \eta \in \mathcal{C} \), \( \langle \eta, S\eta \rangle = 8K_1 k_s^2 \pi^2 > 0 \) by Equation (26). So, \( \langle \chi, S\chi \rangle \geq 0 \).

By Lemma 3, the unconstrained index \( I_u \) equals the number of eigenvectors for which \( \langle \zeta, S\zeta \rangle \) is negative, which is given by the third column of Table 1.

### 7 The constraint matrix

In order to use Equation (10) to compute the index \( I \), we compute the matrix \( \mathbf{W} = (\langle \eta_i, T_j \rangle) \), where \( \eta_i \) is a solution to \( S\eta_i = T_i \), \( \eta_i(0) = \eta_i(1) = 0 \) (see (8) for the definition of \( T_i \)).
One may easily verify that the following are solutions (for $\hat{w} \neq -1$):

$$\eta_1(s) = \frac{1}{8n^2\pi^2 K_3(1 + \hat{w})} \begin{bmatrix} 1 - \cos(2n\pi s) \\ 0 \\ \sin(2n\pi s) \end{bmatrix}$$

$$\eta_2(s) = \frac{1}{8n^2\pi^2 K_1} \begin{bmatrix} 0 \\ 1 - \cos(2n\pi s) \\ 0 \end{bmatrix}$$

$$\eta_3(s) = \frac{1}{8n^2\pi^2 K_1} \begin{bmatrix} 0 \\ -\sin(2n\pi s) \\ 0 \end{bmatrix},$$

and then compute directly that:

$$W = \begin{pmatrix} \frac{1}{4n^2\pi^2 K_3(1 + \hat{w})} & 0 & 0 \\ 0 & \frac{1}{8n^2\pi^2 K_1} & 0 \\ 0 & 0 & \frac{1}{8n^2\pi^2 K_1} \end{pmatrix}$$

Note that $W$ has no zero eigenvalues; it has no negative eigenvalues if $\hat{w} > -1$ (or $\hat{\omega} > 2n\pi(1 - \gamma)$), and one negative eigenvalue if $\hat{w} < -1$ (or $\hat{\omega} < 2n\pi(1 - \gamma)$). Recall that for $\gamma > \rho$, $2n\pi(1 - \gamma) = \hat{\omega}^{n}\rho^{-}$, while for $\gamma < \rho$, $2n\pi(1 - \gamma) = \hat{\omega}^{n}\rho^{+}$. (For $\hat{w} = -1$, there is no solution to $S\eta_1 = T_1$, $\eta_1(0) = \eta_1(1) = 0$, so $W$ would consist of only the lower two-by-two subblock of the above matrix, and would have no negative eigenvalues.)

8 Results

8.1 Index as a function of $\hat{w}$

In Table 2, we compute the index $I$ as a function of $\hat{w}$. The center two columns of the table are derived in the previous two sections, and the final column is found by applying Eq. (10).
<table>
<thead>
<tr>
<th>Range of ( \hat{u} )</th>
<th>( I_u )</th>
<th>( d^{uv}(W) )</th>
<th>( I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^{n+2}+1 &lt; \hat{u} \leq n^{n+3}+1 )</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>( n^{n+1}+1 &lt; \hat{u} \leq n^{n+2}+1 )</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( n^{n-1}+1 &lt; \hat{u} \leq n^{n+1}+1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n^{n-2}+1 &lt; \hat{u} \leq n^{n-1}+1 )</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( n^{n-3}+1 &lt; \hat{u} \leq n^{n-2}+1 )</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>( n^{n+2} &lt; \hat{u} \leq n^{n+3}+1 )</td>
<td>2n-4</td>
<td>0</td>
<td>2n-4</td>
</tr>
<tr>
<td>( n^{n+1} &lt; \hat{u} \leq n^{n+2}+1 )</td>
<td>2n-2</td>
<td>0</td>
<td>2n-2</td>
</tr>
<tr>
<td>( n^{n-1} &lt; \hat{u} \leq n^{n+1}+1 )</td>
<td>2n-1</td>
<td>0</td>
<td>2n-1</td>
</tr>
<tr>
<td>( n^{n-2} &lt; \hat{u} \leq n^{n-1}+1 )</td>
<td>2n</td>
<td>1</td>
<td>2n-1</td>
</tr>
<tr>
<td>( n^{n-3} &lt; \hat{u} \leq n^{n-2}+1 )</td>
<td>2n+2</td>
<td>1</td>
<td>2n+1</td>
</tr>
<tr>
<td>( n^{n+3} &lt; \hat{u} \leq n^{n+4}+1 )</td>
<td>2n+4</td>
<td>1</td>
<td>2n+3</td>
</tr>
</tbody>
</table>

Table 2: Constrained index \( I_u, d^{uv}(W) \), and index \( I \) as a function of intrinsic curvature \( \hat{u} \), broken down by relative sizes of \( \gamma \) and \( \rho \).

Note that for \( \gamma > \rho \), the index does not change at \( \hat{u}^{n,0}+ \), jumps by one at \( \hat{u}^{n,0}+ \), and jumps by two at all other \( \hat{u}^{n,k} \). For \( \gamma < \rho \), the index does not change at \( \hat{u}^{n,0}+ \), jumps by one at \( \hat{u}^{n,0}+ \), and jumps by two at all other \( \hat{u}^{n,k} \). (Recall that for \( \gamma = \rho \), the points \( \hat{u}^{n,0}_{-} \) and \( \hat{u}^{n,0}_{+} \) merge, and otherwise the index changes are as just described). If we further recall that for \( \gamma > \rho \), \( \hat{u}^{n,0}_{+} = 2n\pi(1-\rho) \) and for \( \gamma < \rho \), \( \hat{u}^{n,0}_{-} = 2n\pi(1-\rho) \), we can arrive at a simplified table of the index \( I \) valid for all \( \gamma, \rho \), as shown in Table 3.
\begin{table}
<table>
<thead>
<tr>
<th>$\hat{u}$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{u}^{n+2} &lt; \hat{u} \leq \hat{u}^{n+3}$</td>
<td>4</td>
</tr>
<tr>
<td>$\hat{u}^{n+1} &lt; \hat{u} \leq \hat{u}^{n+2}$</td>
<td>2</td>
</tr>
<tr>
<td>$\hat{u}^{n-1} \leq \hat{u} \leq \hat{u}^{n+1}$</td>
<td>0</td>
</tr>
<tr>
<td>$\hat{u}^{n-2} \leq \hat{u} &lt; \hat{u}^{n-1}$</td>
<td>2</td>
</tr>
<tr>
<td>$\hat{u}^{n-3} \leq \hat{u} &lt; \hat{u}^{n-2}$</td>
<td>4</td>
</tr>
<tr>
<td>$\hat{u}^{n+1} \leq \hat{u} &lt; \hat{u}^{n+2}$</td>
<td>$2n - 4$</td>
</tr>
<tr>
<td>$2n\pi(1 - \rho) \leq \hat{u} &lt; \hat{u}^{n+1}$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>$\hat{u}^{n+1} \leq \hat{u} &lt; 2n\pi(1 - \rho)$</td>
<td>$2n - 1$</td>
</tr>
<tr>
<td>$\hat{u}^{n+2} \leq \hat{u} &lt; \hat{u}^{n+1}$</td>
<td>$2n + 1$</td>
</tr>
<tr>
<td>$\hat{u}^{n+3} \leq \hat{u} &lt; \hat{u}^{n+2}$</td>
<td>$2n + 3$</td>
</tr>
</tbody>
</table>

Table 3: Index $I$ as a function of intrinsic curvature $\hat{u}$. 

These results are displayed graphically in Figure 1, where we show the index as a function of $n$, $\hat{u}$, $\rho$, and $\gamma$.

These results provide a generalization of the results reported for singly-covered circles in the dynamic stability study by Goriely and Shipman [4]. In this dynamics setting, it is the growth of normal modes about the stationary circular solutions that is of interest. Due to the circular symmetry, each vibrational mode is actually a member of a continuous circular family of vibrational modes, corresponding to a choice of orientation of the vibration on the circular rod. This symmetry is reflected here in the fact that the index jumps by 2 at most $\hat{n}^{\pm}$, since any downward direction in the strain energy density surface is actually a member of two-dimensional plane of such downward directions. The exception to this phenomenon occurs at $\hat{u} = 2n\pi(1 - \rho)$, where the index jumps by only 1. By the above argument, the mode associated with this bifurcation point is not a member of a circular family of modes. Based on the computations in the following section, we speculate that this mode corresponds to a torsional mode, such as rotation of the rod about its own centerline.

### 8.2 Bifurcation diagram

Figure 2 shows the bifurcation diagram for ring equilibria as a function of $\hat{u}$. Each point in the diagram represents a ring equilibrium, i.e. a solution of (4) subject to boundary conditions (2). For each equilibrium, the values of $\hat{u}$ and $\int_0^1 u_1(s) \, ds$ are plotted. The linestyle and accompanying numbers on the branches indicate the index of the equilibrium. The horizontal branches represent the circular equilibria from Eq. (5); note that the index on these branches verifies the results in Table 3.

In addition, we show portions of the branches bifurcating from the horizontal branches at the bifurcation points $\hat{n}^{\pm}$. In general, these bifurcating branches correspond to non-circular equilibria; for example, solutions near $\hat{u}^{1\pm}$ are potato-chip-shaped, while solutions near $\hat{u}^{2\pm}$ are non-planar.
figure-eights. On the other hand, the branches emanating from the bifurcation points $2n\pi(1 - \rho)$ correspond to circular equilibria, though they are rotated out of the $y - z$ plane in which the circular equilibria in Eq. (5) sit.

Note that the branch bifurcating from $\hat{u}^{12;+}$ connects to the branch bifurcating from $\hat{u}^{32;+}$. Other connectivities of this type are also present (including $\hat{u}^{23;+}$ to $\hat{u}^{43;+}$ and $\hat{u}^{13;+}$ to $\hat{u}^{53;+}$) but we have not yet classified them. Note also that the branches bifurcating from $\hat{u} = 2n\pi(1 - \rho)$ contain index changes suggesting the existence of secondary bifurcations.

In producing Figure 2, we used the parameter continuation package AUTO [2] to compute the equilibria, and the conjugate point algorithm described in [9] to compute the index (thus, this serves as an independent verification of the index results in Table 3). A comment about these computations is in order. A direct implementation of this problem within AUTO will allow the tracking of the horizontal branches, but not the detection of the bifurcation points and subsequent tracking of the bifurcating branches, because the index changes by 2 at most bifurcation points, and this means that the determinant of the underlying Jacobian used by AUTO to detect bifurcations does not change signs at the bifurcation point, and hence the bifurcation point is not detected. (The bifurcation points $2n\pi(1 - \rho)$ where the index changes by 1 can be detected, and the bifurcating branches followed). The bifurcating branches were instead computed by using a perturbation expansion of the equilibrium equations about $\hat{u} = \hat{u}^{n;+}$, so that the computation could begin directly on the bifurcating branches.

### 8.3 Conditions for stable $n$-covered circles for an intrinsically straight rod

As a final application, we derive from Table 3 the conditions under which an $n$-covered circle is stable for an intrinsically straight rod; in other words, we determine the conditions under which $\hat{u} = 0$ lies in the index 0 region.

**Theorem 1:** If $\hat{u} = 0$, the 1-covered circle has index 0 if and only if $\rho \geq 1$

**Proof:**

For $n = 1$, the index-0 region is $2\pi(1 - \rho) \leq \hat{u} < \hat{u}^{12;+}$. Observe that the upper limit of this index-0 region is greater than 0:

$$\hat{u}^{12;+} = (2 - \gamma - \rho)\pi + \pi \sqrt{(\gamma - \rho)^2 + 16\gamma \rho}$$

$$> (2 - \gamma - \rho)\pi + \pi \sqrt{(\gamma - \rho)^2 + 4\gamma \rho} = (2 - \gamma - \rho)\pi + \pi \sqrt{(\gamma + \rho)^2} = 2\pi > 0.$$

So, if $\rho \geq 1$, $\hat{u} = 0$ is in the index-0 region, while if $\rho < 1$, $\hat{u} = 0$ is not in the index-0 region. □

**Theorem 2:** For $n > 1$ and $\hat{u} = 0$, the $n$-covered circle has index 0 if and only if $\gamma > 1$, $\rho > 1$, and

$$\frac{n - 1}{n} \leq \sqrt{\frac{\gamma - 1}{\gamma} \cdot \frac{\rho - 1}{\rho}}.$$
Proof:

From Table 3, the \( n \)-covered circle with \( \hat{u} = 0 \) has index 0 if and only if \( \hat{u}^{n,n-1,1,-} \leq 0 \leq \hat{u}^{n,n+1,1,+} \). Further, since \( 2n\pi = \hat{u}^{n,n,1,+} < \hat{u}^{n,n+1,1,+} \), we may say more simply that the \( n \)-covered circle with \( \hat{u} = 0 \) has index 0 if and only if \( \hat{u}^{n,n-1,1,+} \leq 0 \). Applying the definition of \( \hat{u}^{n,n-1,1,+} \), we see this is equivalent to

\[
(2 - \gamma - \rho)n\pi + \pi\sqrt{n^2(\gamma - \rho)^2 + 4\gamma\rho(n-1)^2} \leq 0,
\]

or

\[
\sqrt{n^2(\gamma - \rho)^2 + 4\gamma\rho(n-1)^2} \leq (\gamma + \rho - 2)n.
\]

This holds if and only if \( \gamma + \rho - 2 > 0 \) and

\[
\begin{align*}
    n^2(\gamma - \rho)^2 + 4\gamma\rho(n-1)^2 &\leq (\gamma + \rho - 2)^2n^2, \\
    n^2(\gamma^2 - 2\gamma\rho + \rho^2) + 4\gamma\rho(n-1)^2 &\leq (\gamma^2 + \rho^2 + 4 - 4\gamma - 4\rho + 2\gamma\rho)n^2, \\
    4\gamma\rho(n-1)^2 &\leq (4 - 4\gamma - 4\rho + 4\gamma\rho)n^2, \\
    \frac{(n-1)^2}{n^2} &\leq \frac{4\gamma\rho}{4 - 4\gamma - 4\rho + 4\gamma\rho} \\
    \frac{(n-1)^2}{n^2} &\leq \frac{(\gamma - 1)(\rho - 1)}{\gamma\rho}.
\end{align*}
\]

This inequality clearly can not hold if \( \gamma - 1 \) and \( \rho - 1 \) have opposite signs. Combining this with the fact that \( \gamma + \rho - 2 > 0 \) leads immediately to the stated result. \( \square \)

As seen in Figure 1, the set described in Theorem 2 is bounded below by a hyperbola. The equation of this hyperbola is

\[
\gamma = \frac{n^2}{2n-1} - \frac{n^2(\gamma-1)^2}{(2n-1)^2},
\]

which has asymptotes \( \gamma = \frac{n^2}{2n-1} \) and \( \rho = \frac{n^2}{2n-1} \).

Note that in the isotropic case \( (\rho = 1) \) of common interest, these results confirm the known results that the 1-covered circle is stable for all \( \gamma \) and the \( n \)-covered circle for \( n > 1 \) is unstable for all \( \gamma \).

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References


\( \dot{u} = 0 \)
\( \dot{u} = \pi \)
\( \dot{u} = 2\pi \)
\( \dot{u} = 4\pi \)
\( \dot{u} = 6\pi \)

\[ n = 1 \quad n = 2 \quad n = 3 \]

Figure 1: Index of the \( n \)-covered circular equilibrium as a function of \( \gamma \) and \( \rho \) for several values of \( \dot{u} \) and \( n \). When \( \dot{u} = 2n\pi \), the intrinsic shape is an \( n \)-covered circle, so the \( n \)-covered circular equilibrium is always stable (the index is always 0). For other values of \( \dot{u} \), there is always a region of index-0 for \( \gamma \) and \( \rho \) large, while other stiffnesses give positive index.

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Figure 2: Bifurcation diagram of ring equilibria for $\gamma = 2$ and $\rho = 4$: for each equilibrium, we plot the integral of the bending strain $u_1$ versus the intrinsic bending strain $\dot{u}$. Horizontal curves correspond to the $n$-covered circular equilibria. At the bifurcation points $\dot{u}^{n+/-}$, branches of other equilibria (not generally circular) connect to these horizontal branches. The index of each equilibrium is indicated by linestyle (solid for index 0, dashed for index 1, dot-dashed for index 2, dotted for index 3 or more) and also by a superimposed number.