Last time: for indistinguishable particles, we must have \( \hat{P}_{12} \Psi(\vec{r}_1, \vec{r}_2) = \pm \Psi(\vec{r}_1, \vec{r}_2) \).

However, our solution for distinguishable particles,
\[ \Psi(\vec{r}_1, \vec{r}_2) = \Psi_\alpha(\vec{r}_1) \Psi_\beta(\vec{r}_2) \]
doesn't fulfill this requirement.

Solution: symmetrize or antisymmetrize \( \Psi \):
\[
\Psi_s(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} \left[ \Psi_\alpha(\vec{r}_1) \Psi_\beta(\vec{r}_2) + \Psi_\beta(\vec{r}_1) \Psi_\alpha(\vec{r}_2) \right]
\]
\( \hat{P}_{12} \Psi_s = + \Psi_s \)

OR
\[
\Psi_A(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} \left[ \Psi_\alpha(\vec{r}_1) \Psi_\beta(\vec{r}_2) - \Psi_\beta(\vec{r}_1) \Psi_\alpha(\vec{r}_2) \right]
\]
\( \hat{P}_{12} \Psi_A = - \Psi_A \)

Using relativistic quantum mechanics, one can prove the

**Spin-Statistics Theorem:**

All particles with half-integer spin (e.g., electrons) must have antisymmetric multiparticle wavefunctions. They are called "Fermions".

All particles with integer spin (e.g., photons) must have symmetric multiparticle wavefunctions. They are called "Bosons".

Immediate consequence:

**Pauli Exclusion Principle:**

Two Fermions cannot occupy the same state.

(If they did occupy the same state, we'd have \( \Psi_A = \frac{1}{\sqrt{2}} \left[ \Psi_\alpha(\vec{r}_1) \Psi_\beta(\vec{r}_2) - \Psi_\beta(\vec{r}_1) \Psi_\alpha(\vec{r}_2) \right] = 0 \).

Now: to include spin

For Fermions, we still need \( \hat{P}_{12} \Psi = - \Psi \), where now \( \Psi \) includes the spin degrees of freedom. There are two ways to achieve this:

**Aspace:** Antisymmetrize the spatial part.

\[
\Psi_A = \frac{1}{\sqrt{2}} \left[ \Psi_\alpha(\vec{r}_1) \Psi_\beta(\vec{r}_2) - \Psi_\beta(\vec{r}_1) \Psi_\alpha(\vec{r}_2) \right]
\]

\( [X_\alpha(1) X_\beta(2) + X_\beta(1) X_\alpha(2)] \)

**Aspin:** Antisymmetrize the spin part.

\[
\Psi_A = \frac{1}{\sqrt{2}} \left[ \Psi_\alpha(\vec{r}_1) \Psi_\beta(\vec{r}_2) + \Psi_\beta(\vec{r}_1) \Psi_\alpha(\vec{r}_2) \right]
\]

\( [X_\alpha(1) X_\alpha(2) - X_\beta(1) X_\beta(2)] \)
Generalized Pauli exclusion principle:

If we've chosen Aspace, then we must have $\alpha \neq \beta$ else $\Phi_A \to 0$.
If we've chosen Aspin, then we must have $\alpha = \beta$ else $\Phi_A \to 0$.

Any two Fermions must either occupy two distinct spatial states or two distinct spin states (or both).

Example 1: $\alpha = \beta = 15 \Rightarrow$ must choose Aspin $\Rightarrow \alpha = +, \beta = -$.

Example 2: $\alpha = 15, \beta = 25 \Rightarrow$

- Either choose Aspin $\Rightarrow \alpha = +, \beta = -$ "singlet state"
  - Book pp. 271-3 $\Rightarrow$ the total spin angular momentum for this state is zero
  - $\Rightarrow$ the spins are exactly antiparallel

- Or choose Aspace
  - Three options: $\alpha = +, \beta = +, \Phi \propto X_+(1)X_+(2)$ "Both up"

  

  $a = -, \beta = -, \Phi \propto X_-(1)X_-(2)$ "Both down"

  

  $a = +, \beta = -, \Phi \propto [X_+(1)X_-(2) + X_-(1)X_+(2)]$ "Both sideways"