Coupled oscillators

General solution:
A sum of pendulum & breathing modes.

To find $x_1(t)$ and $x_2(t)$ given initial conditions:

1) Use initial conditions to find $A_p, \phi_p$; $A_b, \phi_b$

2) $S_p = A_p \cos (\omega_p t + \phi_p)$
   $S_b = A_b \cos (\omega_b t + \phi_b)$

3) $x_1 = \frac{1}{\sqrt{2}} (S_p + S_b)$
   $x_2 = \frac{1}{\sqrt{2}} (S_p - S_b)$

Example

Initial velocities zero $\Rightarrow S_p$ & $S_b$ are initially at max or min
$\Rightarrow \phi_p = \phi_b = 0$ (amplitudes can be negative if necessary)

$S_p = A_p \cos \omega_p t$ $\Rightarrow S_{p0} = A_p$
$S_b = A_b \cos \omega_b t$ $\Rightarrow S_{b0} = A_b$

$S_{p0} = \frac{x_{10} + x_{20}}{\sqrt{2}} = \frac{A}{\sqrt{2}}$
$S_{b0} = \frac{x_{10} - x_{20}}{\sqrt{2}} = \frac{A}{\sqrt{2}}$
Example of finding $x_1(t)$ and $x_2(t)$

Initial conditions:

$\dot{x}_{10} = x_{20} = 0$

$\dot{x}_{10} = \frac{d}{\sqrt{2}} \quad x_{20} = 0$

1) Find $A_p, \varphi_p, A_b, \varphi_b$

Initial velocities zero $\Rightarrow \varphi_p = \varphi_b = 0$

$A_p = S_{p0}$

$A_b = S_{b0}$

$S_{p0} = \frac{1}{\sqrt{2}} (x_{10} + x_{20}) = \frac{d}{\sqrt{2}}$

$S_{b0} = \frac{1}{\sqrt{2}} (x_{10} - x_{20}) = \frac{d}{\sqrt{2}}$

2) $S_p = A_p \cos(\omega pt + \varphi_p)$

$S_b = A_b \cos(\omega bt + \varphi_b)$

$\Rightarrow$ For this case

$S_p = \frac{d}{\sqrt{2}} \cos \omega pt$

$S_b = \frac{d}{\sqrt{2}} \cos \omega bt$

3) $x_1 = \frac{1}{\sqrt{2}} (S_p + S_b)$

$x_2 = \frac{1}{\sqrt{2}} (S_p - S_b)$

$\Rightarrow$ For this case,

$x_1 = \frac{d}{2} (\cos \omega pt + \cos \omega bt)$

$x_2 = \frac{d}{2} (\cos \omega pt - \cos \omega bt)$

The complicated beating behavior we observed in the demonstration for $x_1(t)$ and $x_2(t)$ is really just the superposition of two sinusoidal motions!

Hilbert Space: An efficient way to find $A_p, \varphi_p, A_b, \varphi_b$

- Absolutely central to quantum mechanics

There are many types of Hilbert space. Here's the relevant one for our situation:

Pendulum: $x_{10} = x_{20}$

Total coord along pendulum axis

$= \frac{1}{\sqrt{2}} (x_{10} + x_{20}) = S_{p0}$

Total coord along breathing axis

$= \frac{1}{\sqrt{2}} (x_{10} - x_{20}) = S_{b0}$

Breathing: $x_{20} = -x_{10}$
So, we could think of the rotated axes as the $S_{po}$ & $S_{bo}$ axes:

\[ |e_p\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle) \]

\[ |A\rangle = (A_1, A_2) \]

an "eigenvector" -- a vector in Hilbert space corresponding to one of the simple, characteristic behaviors of the system.

To determine the components, we use dot products:

\[ A_x = \hat{i} \cdot \hat{A} \]

\[ A_y = \hat{j} \cdot \hat{A} \]

To find the components of $|A\rangle$ along the $S_{po}$ & $S_{bo}$ axes, we need the equivalent of the dot product of $|e_p\rangle & |A\rangle$ and the dot product of $|e_b\rangle & |A\rangle$. 