The expansion of the universe

In assignment #3 and in the subsequent class, we investigated the Big Bang Theory of the universe. One important result was that if the average density of the universe is less than the “critical density” ($\rho_{\text{crit}}$) then the universe will continue to expand forever. Alternatively, if the density of the universe is greater than $\rho_{\text{crit}}$, then the universe will eventually stop expanding, begin collapsing, and will end in a “big crunch”. Therefore a key question in cosmology is: “What is the average density of the universe?” The best current observations indicate that the average matter density is about 0.3 $\rho_{\text{crit}}$ even if we include dark matter. If there is significantly more matter than this, then the universe would have more large-scale structure than it appears to have (see Rees).

In the last decade, two important astronomical observations challenge the hypothesis that we live in a universe with $\rho_{\text{total}} = \rho_{\text{matter}} = 0.3 \rho_{\text{crit}}$:

**Cosmic Microwave Background (CMB):** The cosmic microwave background shows that the universe radiates like a near-perfect blackbody with small deviations from isotropy. The angular size of the most prominent of these fluctuations provides a fairly direct measurement of the average density of the universe. Detailed measurements of the CMB made in 1999 by experiments like BOOMERanG and MAXIMA, and then confirmed with higher fidelity by WMAP in the first decade of the 21st century, showed that the average total density of the universe equals $\rho_{\text{crit}}$. I just told you that the average matter density is only 0.3 $\rho_{\text{crit}}$. So where is the other 70% of the density of the universe?

**Hubble’s Law far away and a long time ago:** Studying deviations from the Hubble Law ($v = H_0 d$) at large distances from us (i.e. far into our universe’s past) is a direct way to learn about the large-scale geometry and history of the universe. We observe the universe to be expanding now; if the density in the universe is all contributed by gravitating matter, then the expansion of the universe must decelerate owing to the gravitational attraction of the matter inside of it. In parallel with observations of the CMB that revealed $\rho_{\text{universe}} = \rho_{\text{crit}}$, observations of the recessional velocities of distant Supernova Ia revealed that the expansion of the universe is actually accelerating. How can we reconcile observations of the CMB with observations of the expansion rate of the universe? There is even more matter than we can see, yet the expansion of the universe is accelerating.

There is a way out of this dilemma. If one can conjure up a form of “matter” that has positive mass density but a negative pressure (which will be a source of negative gravity), then we can both account for the missing 70% of the matter in the universe and account for its accelerating expansion. This form of density actually
exists in the ordinary world. The electromagnetic field has a positive energy density (and equivalent mass density if we divide by \( c^2 \)) and a negative pressure. You may see the latter property of electric fields this semester in Physics 106 (or if not, then in Physics 309 if you take that class). This negative pressure is one way to explain why conductors are pulled into an electric field. However, electromagnetic fields couldn’t possibly be the cosmological source we seek. They would have been detected already.

Astrophysicists call the unaccounted source of mass (and negative gravity) “dark energy” to indicate that it is unlike ordinary matter and dark matter. Dark energy refers to the energy density of empty space. In this assignment, we’ll investigate some properties of a universe that is dominated by negative gravity. We will use many of the same arguments as in Assignment #3, so you should rely on that assignment and the accompanying solutions.

Einstein himself introduced one form of dark energy 80 years ago. He called it the cosmological constant. The effective mass density associated with the cosmological constant, \( \rho \Lambda \), is a constant even while the universe is expanding. This means that the total effective mass of this dark energy is not conserved, but is always increasing! This mass is a source of positive gravity just like any other form of mass. However, the negative pressure of this form of dark energy generates exactly 3 times as much negative gravity than positive gravity. For our purposes, we can say that the net gravitational mass density of the cosmological constant is just \(-2\rho \Lambda\).

Let’s begin by considering a universe filled by only this form of mass-energy. Consider the motion of a distant galaxy at distance \( r \) from us. Its recessional velocity is given by the Hubble Law. Let’s solve for the subsequent motion of this galaxy. Recall the argument in Part c of Assignment 3 - only the mass inside a sphere of radius \( r \) (with us at the center) will exert a gravitational force on the galaxy, and that force will be as though all the mass within the sphere is located at the center.

1. Give an expression for the gravitational force on the galaxy in terms of \( \rho \Lambda \), \( G \), \( r \), and \( m \) (the mass of the galaxy). Include all numerical constants - i.e. we are not yet doing this as a back-of-the-envelope calculation. Remember that the effective gravitational mass density is given by \(-2\rho \Lambda\). Be sure to indicate the direction of the force, a minus sign indicates attraction and a plus indicates repulsion. Now write down the equation of motion of this galaxy, \( F = ma = m \frac{d^2 r}{dt^2} \).

2. The equation in Problem 1 is a second order, linear differential equation. Let’s solve it using a method similar to that in Problem b on Assignment 3. Begin by finding the potential energy, \( U(r) \), of the repulsive gravitational force. If you can’t remember how to get from force to potential energy then look it up in your
physics textbook. Remember that $\rho_\Lambda$ is constant, not the mass within the sphere of radius $r$, so you must express the force in terms of $\rho_\Lambda$ and not $M$. The potential energy should be negative. Pick your constant of integration so that $U(0) = 0$.

3. Write an expression for the total energy $E$ (potential plus kinetic) of the galaxy. Show that if $\rho_\Lambda = \rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}$, then the total energy is equal to 0. Remember that CMB measurements indicate that the average density of the universe is equal to $\rho_{\text{crit}}$.

4. Since energy is conserved, $E$ is always equal to 0 and you can solve for $v = dr/dt$ as a function of $r$. Solve this equation for $r(t)$ as a function of time, $t$. Pick the constant of integration so that $r = r_0$ at $t = t_0$, the current age of the universe. Express your answer in terms of $r_0$, $H_0$, $t$, and $t_0$. Compare this solution with that of a universe filled with ordinary matter with the critical density (see Problem g on Assignment #3).

5. Show that the Hubble constant at any time in the future or past, i.e. $H = \frac{dr}{dt} \frac{1}{r}$ is the same as the current Hubble constant, $H_0$. Remember this was not the case for the critical density universe filled with ordinary matter.

6. Another way to solve for the motion of a galaxy is directly from the differential equation of motion in a) above. This will be especially helpful to you if you were not able to complete b), c), and d) above. One tried and true method of solving differential equations is simply to “guess” the functional form of the solution. In this case, guess a solution of the form $Ae^{pt}$ where $A$ and $p$ are constants. Now just plug into the equation of motion and solve for the values of $A$ and $p$!

Now we’re going to see qualitatively how it is that the supernovae teams were able to distinguish between a matter and a dark energy dominated universe. First, some astronomy.

7. One of the tried and true ways astronomers measure distances is by the standard candle method. If you know the intrinsic luminosity of a source, then you can determine its distance by measuring its apparent brightness. What is the quantitative relationship between the luminosity (power) emitted by a source and the flux (power per area) that the source is observed to have by an observer at a distance, $d$ from the source?

8. Lets assume that all galaxies have the same luminosity $L$. Using your answer to 7, you can deduce the distance to any galaxy from its observed flux $F$ at the Earth. Astronomers often plot data on a logarithmic scale (usually base 10), because of the large range spanned by many astronomical quantities. Make a sketch of a plot of the log of the flux, $F$, versus the log of the distance to a galaxy. You will see that the units of flux and distance won’t change the shape of the plot but rather just displace the curve up and down or right and left. Therefore I want
you to graph it carefully (use graph paper or a computer) so that the slope of the curve is accurate.

9. Now make a plot of the Hubble law, but this time plot the log of the flux, \( \log(F) \), versus the log of the velocity, \( \log(v) \). Again, make your plot carefully so that the slope of the curve is accurate (disregard the units of flux and velocity).

10. When we observe very distant galaxies, we see them as they appear in the past since it takes the light from them a great time to reach us. The plot you made in 9 was therefore only true for a universe in which galaxies’ velocities don’t change with time, i.e. a universe with very little mass or energy of any kind. If in the past galaxies were receding at higher velocity than they are today, then the most distant galaxies should not obey the Hubble law plotted in 9. Make \( \log(F) \) versus \( \log(v) \) plots of three universes: i) a universe with very little energy and matter [the same as 9. above], ii) the critical density matter dominated universe of Assignment #3, and iii) the cosmological constant dominated universe of Problem 4 above. Put all three curves on the same plot. Just make qualitative sketches. Its this type of figure that the high-redshift supernova teams use to argue that the universe’s expansion is accelerating. NOTE: This problem is actually a bit more complicated than we’ve treated it here, because at the large distances where the effect of accelerating expansion is noticeable, the curvature of space-time must be taken into account using general relativity.

11. Now lets be quantitative. Suppose the light we are now receiving from a distant galaxy was emitted at a time \( t = \frac{1}{3H_0} \) ago, i.e. at a time \( t - t_0 = -\frac{1}{3H_0} \). Compute the velocity of the galaxy at that time in terms of its current velocity. Do this both for the critical density matter dominated universe of Assignment #3 and for the cosmological constant dominated universe of Problem 4 above. Express \( v/v_0 \) as a decimal fraction. This will give you an indication of the magnitude of the effect that the supernovae teams have claimed to detect. We will discuss the accuracy of the observations in class.

One of the topics of Chapter 8 of Rees is the formation of structures in the universe, and it is to this topic that we now turn. Just consider the actual matter density in the universe for these following three parts. You may remember that the very first workshop problem you considered in the first class this semester was, “How cold does a gas cloud have to be before it will begin to collapse to form stars?” In the following, you’re going to consider this problem again in a different context. Lets consider the size of the structures that began to form after the universe had cooled enough for neutral hydrogen to form, i.e. when the hydrogen was no longer ionized. This was referred to in Rees and occurred when the universe was at a temperature of about 3000 K and was about 1/1100 its present linear size (i.e. everything was 1100 times closer than it is now).
12. If the current (ordinary) matter density is \(0.04 \rho_{\text{crit}}\), what was the matter density (in kg/m\(^3\)) at that epoch?

13. Now the same back-of-the-envelope argument we used in the first class to determine the smallest sized objects that could begin to collapse. Give both their mass and radius (in solar masses and pc). Recall that the essence of the argument was that the gravitational force has to be great enough to overcome gas pressure. The quantities you just calculated are called the \textit{Jeans mass} and the \textit{Jeans length}.

14. About how long did it take such an object to collapse (to form stars or star clusters or whatever)? Remember that collapse time was the first back-of-the-envelope calculation we did in class. It actually takes somewhat longer than you will estimate if you neglect the expansion of the universe (as I did when I worked the problem).