White dwarfs, neutron stars, and black holes

Read sections in your physics text about gravity (e.g. potential energy, escape velocity, the virial theorem, etc.). If covered, also read the sections about the Pauli Exclusion Principle and the Heisenberg Uncertainty Principle. Read the article on active galaxies and quasars at the NASA Web site

http://imagine.gsfc.nasa.gov/docs/science/know_l1/active_galaxies.html

(Note: l1 = lowercase L followed by the number one).

To investigate this week’s topic, we will use two implications of quantum theory: the Pauli Exclusion Principle and the Heisenberg uncertainty principle.

For an astrophysical object to exist in a stable configuration (e.g. to neither expand nor contract), there needs to be a balance between the gravitational force pushing in on the material and some interior pressure pushing out on the material. During a star’s life, the fusion of nuclear fuel in its core provides radiation and gas pressure that supports the star against gravity. When a star expends all of this fuel, gas pressure is no longer able to withstand gravity and the star begins to collapse. Eventually, at high enough densities, another force is able to withstand gravity: the degeneracy force provided by the electrons in the star.

**Electron degeneracy pressure -** This pressure is a direct consequence of the Pauli exclusion principle. Electrons are spin 1/2 elementary particles called fermions. No two fermions can be in the same quantum state at the same time. This is the essence of the Pauli exclusion principle and is the reason why successive electrons in atoms occupy higher and higher energy levels. Another consequence of this principle is that the electrons very dense material do their best to avoid each other, each effectively occupying a volume \( \Delta V_{\text{electron}} \sim 3 \Delta x^3 \sim \frac{V_{\text{star}}}{N_e} \) where \( V \) is the volume of the star and \( N_e \) is the number of electrons. The ever decreasing volume available for electrons to occupy as a star collapses results in an electron pressure, thanks to the Heisenberg uncertainty principle.

**The Heisenberg uncertainty principle -** A particle’s position, \( x \), and its momentum, \( p \), cannot be known to infinite precision. In fact:

\[
\Delta p \Delta x \geq \frac{\hbar}{2} \tag{1}
\]

\( \hbar \) is Planck’s constant \( (1.05 \times 10^{-34} \text{ Joules-second}) \). As an electron is confined to a volume whose linear dimension is \( \Delta x \) meters, its momentum cannot be less than \( \Delta p \). In a collapsed
star, as density increases ($\Delta x$ decreases), $\Delta p$ must increase. (Up to a stellar mass of $1.4M_\odot$) this electron motion provides enough pressure to support the star against continued collapse.

**Neutron degeneracy pressure** - Above a certain gravitational pressure on a dead star (e.g. beyond a certain total mass of the star), the electron degeneracy pressure is not strong enough to support the star against gravity. In this case, electrons and protons combine to form neutrons (plus a neutrino). The result is a neutron star in which gravity is balanced by neutron degeneracy forces. Neutrons are fermions, so they also obey the Pauli Exclusion Principle. Neutron degeneracy pressure is strong enough to support a dead star's weight, up to a stellar mass of $3M_\odot$.

In Astro 205 and Astr 321 we study the structure of collapsed, or degenerate, stars by computing the pressure resulting from this electron pressure and then require that it balance gravity. We are going to approach this with a more simple order-of-magnitude, back-of-the-envelope calculation.

**The Virial Theorem** - Our final ingredient for now is the virial theorem. This theorem states that for any gravitationally bound system, $<KE> = -0.5 <PE>$. In words, the mean kinetic energy of the particles is equal to minute 1/2 times the mean gravitational potential energy. Therefore, $|KE| \sim |PE|$.

1. Construct a simple model of a cold, collapsed star with mass $M_{\text{star}}$. Derive an approximate relation for its radius in terms of $G$, $\hbar$, $m_e$, $m_H$, and $M_{\text{star}}$. Do this for a star supported by electron degeneracy pressure. (With three masses in this problem, dimensional analysis isn’t a good way to approach this question.)

2. Compute the value of the radius of this collapsed star if its mass is equal to that of the Sun. Also compute the density of this star. What is an object with comparable size to your calculated radius? This type of degenerate star is known as a “white dwarf” and will be our Sun’s fate in about 6 billion years.

3. Now derive an approximate relation for the radius of a neutron star in terms of $G$, $\hbar$, $m_H$, and $M_{\text{star}}$. Notes that the mass of a neutron is nearly the same as the mass of a hydrogen atom.

4. Compute the radius of a neutron star with a mass of $1.4M_\odot$. Also compute the density. What is an object comparable in size to your calculated radius?

5. From 1. and 3., find the ratio of the radius of a neutron star to that of a which dwarf of the same mass. Express your answer in terms of fundamental constants.
6. We will now use classical physics to get a sense of what happens when a degenerate object has $M > 3M_\odot$; This object will be a black hole. Consider an object so small and dense that the escape velocity from its surface is the speed of light. Such an object will appear black, because no light can escape from it. Calculate the critical radius below which an object will appear “black”, in terms of its mass, $M$, $G$, and $c$. This defines the location of a black hole’s Schwarzschild radius. If you kept the factor of two, you’ve derived the exact expression predicted by general relativity (partially a coincidence). $R_{Sch}$ doesn’t represent an actual, physical, surface. It instead represents a horizon from within which no information can be transmitted to the outside world.

7. What is $R_{Sch}$ for a $3M_\odot$ black hole? Express your answer in meters, and compare with the radius you derived for a neutron star. (Don’t forget that the latter was an order-of-magnitude estimate).

8. Supermassive ($10^6 - 10^{10} M_\odot$) black holes reside in the centers of most galaxies. These black holes power the quasars and active galaxies described in the reading. The energy source is the gravitational potential energy released as matter falls into the black hole. If a mass, $m$, falls into a black hole, how much gravitational potential energy is liberated before it falls through the horizon? Express your answer in terms of fundamental constants.

9. Quasars typically emit 100 times more power (energy per second) than the sum of all the $10^{11}$ stars in the host galaxy. Using the expression you derived in the previous question, estimate the amount of matter that must be consumed by a black hole in order to account for a $10^{40}$ Watt (Joules per second) quasar. Express your answer in solar masses per year. Compare this to the $10^{11} M_\odot$ of mass in a typical galaxy.

Comment.