An Introduction to Invariants of Legendrian Knots

Joshua M. Sabloff

Haverford College

AIM Workshop on Legendrian and Transverse Knots
Survey some classical and non-classical invariants of Legendrian knots with an eye towards:
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- Combinatorial definitions and geometric motivations for the invariants
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- Connections between the invariants
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- Combinatorial definitions and geometric motivations for the invariants
- Connections between the invariants
- Extensions of the invariants
1 Classical Invariants
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2. Combinatorial Definitions of Invariants
   - Normal Rulings
   - The Chekanov-Eliashberg DGA
   - Computable Invariants from the DGA
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3. Geometry of the Invariants
   - Generating Families
   - Legendrian Contact Homology
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4. Extensions
Where Are We?

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   Generating Families
   Legendrian Contact Homology

4 Extensions
• Work in the **standard contact 3-space** ($\mathbb{R}^3, \xi_0 = dz - y\,dx$).
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• Use both the **Lagrangian (xy)** and **front (xz)** projections:
We can “resolve” a front diagram into a Lagrangian diagram of an isotopic knot:
The Thurston-Bennequin number $tb$ measures difference between Seifert framing and framing from $\xi$.

$tb = -2$

$tb = \pm 1$
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- $tb$ may be computed from the writhe of the $xy$ diagram.

The rotation number $r$ measures twisting of $K'$ inside a trivialization of $\xi$.

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- ... or writhe $-\#$right cusps in the $xz$ diagram

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- $r$ may be computed from the rotation number of the $xy$ diagram.
- ...or #down cusps $-$ #up cusps in the $xz$ diagram.

- $tb = -2$
- $r = \pm 1$
The classical invariants are restricted by the underlying smooth knot type:

- \(tb(K) + |r(K)| \leq 2g(K) - 1\) [Bennequin]
- \(tb(K) + |r(K)| \leq 2g_s(K) - 1\) [Rudolph]
- And others using \(\tau(K), s(K),\) Khovanov homology, etc.
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- \( tb(K) + |r(K)| \leq \min \deg_a H_K(a, z) - 1 \) [Morton, Franks-Williams, Fuchs-Tabachnikov, ...]
- \( tb(K) \leq \min \deg_a F_K(a, z) - 1 \) [Tabachnikov, Fuchs-Tabachnikov, Chmutov-Goryunov, ...]
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To what extent do the classical invariants classify Legendrian knots in a given knot type?
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- But the classical invariants do not suffice in general

- Still, there are at most finitely many Legendrian knots with a fixed set of classical invariants [Colin-Giroux-Honda]
Geography via “Mountain Ranges”

Unknot [Eliashberg-Fraser]

\[
\begin{array}{ccccccc}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\top & 0 & 1 & 2 & 3 & -1 & -2 & -3 \\
-1 & -2 & -3 & -4 & tb \\
-2 & -3 & -4 & -2 & -3 & -4 \\
-3 & -4 & -3 & -4 & -4 \\
-4 & -4 & -4 & -4 & -4 & -4 \\
\end{array}
\]
Classical Invariants

Geography via “Mountain Ranges”

5₂ Knot [Chekanov, Epstein-Fuchs-Meyer]

\[
\begin{array}{ccccccc}
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
1 & & & & & & \\
0 & & & & & & \\
tb & & & & & & \\
-1 & & & & & & \\
-2 & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]
Classical Invariants

Geography via “Mountain Ranges”

(2,3) Cable of (2,3) Torus Knot [Etnyre-Honda]
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4 Extensions
A **ruling** of a front is a 1-to-1 correspondence between left and right cusps together with two paths joining each pair of cusps such that:

1. The interiors of the paths are disjoint and meet only at the cusps.
2. Any two paths meet only at crossings and cusps.
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Note that fronts of stabilized knots do not have any rulings
We don’t want to allow all types of “switches”:
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Banning these switches gives a normal ruling of a front
Two Examples
We can refine the notion of a normal ruling further by assigning a grading \((\text{modulo } 2r(K))\) to each crossing.
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Define a Maslov potential $\mu$ on each strand and let the grading of each crossing be

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A graded ruling is one whose switches all have grading 0.
We can organize the set of rulings of a front $D$ into a polynomial!

$$\theta(R) = \text{cusps}(D) - \text{switches}(R)$$

$$\Theta_D(k) = \{\text{rulings } R : \theta(R) = k\}$$
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$$\Theta_D(k) = \{\text{rulings } R : \theta(R) = k\}$$

The ruling polynomial of a front $D$ is:

$$R_D(z) = \sum \#\Theta_D(1 - k)z^k$$
Theorem (Chekanov-Pushkar)

The (graded or ungraded) ruling polynomial is a Legendrian invariant.
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Example

The following $5_2$ knots are not Legendrian isotopic.

\[ R(z) = 1 + z^2 \]

\[ R(z) = 1 \]
Theorem (Rutherford)

The ungraded $R_K(z)$ is the coefficient of $a^{-tb(K)-1}$ in $F_K(a, z)$, so the Kauffman bound is sharp if and only if there is a ruling.
This invariant takes the form of a differential graded algebra.
This invariant takes the form of a \textit{differential graded algebra}.

First, the algebra: Label the crossings in an $xy$ diagram with $\{1, \ldots, n\}$. 

\begin{center}
\begin{tikzpicture}
\node[below] at (0,0) {1};
\node[below] at (0,-1) {2};
\node[above] at (1,0) {3};
\node[above] at (-1,0) {4};
\node[below] at (1,-1) {5};
\draw (0,0) to[out=270,in=90] (1,0);
\draw (-1,0) to[out=270,in=90] (0,0);
\draw (0,0) to[out=90,in=270] (0,-1);
\draw (0,-1) to[out=90,in=270] (0,0);
\end{tikzpicture}
\end{center}
This invariant takes the form of a differential graded algebra.

First, the algebra: Label the crossings in an $xy$ diagram with $\{1, \ldots, n\}$.

Let $A$ be the vector space over $\mathbb{Z}_2$ generated by labels $\{q_1, \ldots, q_n\}$, and let $A = \bigoplus_{k=0}^{\infty} A^\otimes k$ be the unital tensor algebra over $A$, graded as for rulings.
To define the differential, first decorate the crossings:
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To find contributions to $\partial q_i$, look for smoothly immersed disks with convex corners whose boundary lies in the knot diagram. There is a $+$ corner at $i$ and $-$ corners elsewhere.

Record the $-$ corners in counterclockwise order:

$$\partial q_1 = q_5 q_4 q_3$$
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\[ \partial q_i \]

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\[ \partial q_1 = q_5 q_4 q_3 + q_5 \]
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To find contributions to $\partial q_i$, look for smoothly immersed disks with convex corners whose boundary lies in the knot diagram. There is a $+$ corner at $i$ and $-$ corners elsewhere.

Record the $-$ corners in counterclockwise order:

$$\partial q_1 = q_5 q_4 q_3 + q_5 + 1 + \cdots$$
The following disk would be counted (as a 1) in the differential:
Theorem (Chekanov)

Let $\partial$ have degree $-1$, $\partial^2 = 0$, and the Legendrian contact homology $H_*(A, \partial)$ (denoted $LCH_*(K)$) is invariant under Legendrian isotopy.

In fact, something more subtle is true: the "stable tame isomorphism" class of $(A, \partial)$ is an invariant. But how do we extract useful information out of this?

Note that this invariant vanishes on all stabilized knots.
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In fact, something more subtle is true: the “stable tame isomorphism” class of \((A, \partial)\) is an invariant. But how do we extract useful information out of this?

Note that this invariant vanishes on all stabilized knots.
Time for a break!
An \textit{augmentation} of \((\mathcal{A}, \partial)\) is an algebra map \(\epsilon : \mathcal{A} \rightarrow \mathbb{Z}_2\) such that:

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3. \(\epsilon\) is **graded** if it has support on generators of grading 0
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There are only finitely many of these!
Augmentation Numbers

We can form an invariant by counting augmentations as follows:

\[ \chi(A) = \sum_{k \geq 0} (-1)^k a_k + \sum_{k < 0} (-1)^{k+1} a_k \]

Now let:

\[ \text{Aug}(A) = \frac{2 + (-1 - \chi(A))}{2 \# \text{augmentations}} \]

Theorem: \( \text{Aug}(A) \) is a Legendrian invariant (now denoted \( \text{Aug}(K) \)).
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- Let $a_k$ be the number of generators of $\mathcal{A}$ of grading $k$, and define:

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**Theorem**

$\text{Aug}(\mathcal{A})$ is a Legendrian invariant (now denoted $\text{Aug}(K)$)
Theorem (Fuchs, Fuchs-Ishkhanov, JMS)

The Chekanov-Eliashberg DGA of a Legendrian knot $K$ has an (graded) augmentation iff a front diagram of $K$ has a (graded) ruling.
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Theorem (Ng-JMS)

$$\text{Aug}(K) = R_K(2^{-\frac{1}{2}})$$
Break up $\partial$:

$$\partial = \partial_0 + \partial_1 + \partial_2 + \cdots,$$

where $\partial_k : A \to A^\otimes k$.
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Given an augmentation $\epsilon$, define an automorphism $\varphi^\epsilon$ of $A$, by:

$$\varphi^\epsilon(q) = q + \epsilon(q)$$
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Given an augmentation $\epsilon$, define an automorphism $\varphi^\epsilon$ of $A$, by:

$$\varphi^\epsilon(q) = q + \epsilon(q)$$

We get a new differential $\partial^\epsilon = \varphi^\epsilon \partial (\varphi^\epsilon)^{-1}$ with $\partial_0^\epsilon = 0$. Then:

$$\partial^2 = 0 \implies (\partial_1^\epsilon)^2 = 0.$$
It follows that \((A, \partial_1^\epsilon)\) is a finite-dimensional chain complex whose homology is denoted \(LCH^\epsilon_*(K)\). Record the homology in a Poincaré-Chekanov polynomial:

\[
P_\epsilon(t) = \sum_{k=-\infty}^{\infty} t^k \dim H_k(A, \partial_1^\epsilon)
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**Theorem (Chekanov)**

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**Example**

This can also distinguish the 5_2’s, as they have linearized invariants \(\{2 + t\}\) and \(\{t^{-2} + t + t^2\}\)
Theorem (JMS)

*For any linearization, \( \dim \text{LCH}^\epsilon_k = \dim \text{LCH}^\epsilon_{-k} \) for \(|k| \neq 1\), while \( \dim \text{LCH}^\epsilon_1 = \dim \text{H}^\epsilon_{-1} + 1 \).*
Theorem (JMS)

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More refined computable invariants can be derived from $(\mathcal{A}, \partial)$:
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More refined computable invariants can be derived from \((\mathcal{A}, \partial)\):

- Cup and Massey product structures on the linearization using \( \partial_k \), \( k \geq 2 \) [Etnyre-JMS et al.]
Theorem (JMS)

For any linearization, \( \dim LCH_k^\varepsilon = \dim LCH_{-k}^\varepsilon \) for \( |k| \neq 1 \), while \( \dim LCH_1^\varepsilon = \dim H_{-1}^\varepsilon + 1 \).

More refined computable invariants can be derived from \((A, \partial)\):

- Cup and Massey product structures on the linearization using \( \partial_k \), \( k \geq 2 \) [Etnyre-JMS et al.]
- The characteristic algebra \( C(A, \partial) = A/\text{im}\partial \) up to isomorphism and adding free generators. [Ng]
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4. Extensions
One way to produce Legendrian curves in the standard contact $\mathbb{R}^3$ (thought of as the 1-jet space $J^1(\mathbb{R})$) is to use graphs of derivatives:

$$x \mapsto (x, f'(x), f(x))$$
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We can obtain “non-graphical” Legendrians by extending our functions to

$$f : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$$

that are nondegenerate (in some sense) and “linear-quadratic at infinity”, i.e. equal to the sum of nondegenerate linear and a nondegenerate quadratic outside a compact set.
Instead of embedding the whole domain, we look at the fiber critical set

\[ \Sigma_f = \{(x, e) \in \mathbb{R} \times \mathbb{R}^N : \partial_e f(x, e) = 0\} \]
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\[ \Sigma_f = \{(x, e) \in \mathbb{R} \times \mathbb{R}^N : \partial_{e} f(x, e) = 0\} \]

Then embed \( \Sigma_f \to \mathbb{R}^3 \) as before:

\[(x, e) \in \Sigma_f \mapsto (x, \partial_x f, f(x, e))\]
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Theorem (Pushkar?, Fuchs-Rutherford)

Any Legendrian knot with a normal ruling can be obtained this way.
An Example
A generating family \( f : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) induces a normal ruling as follows:

- For all but finitely many fixed \( x \), \( f_x : \mathbb{R}^N \rightarrow \mathbb{R} \) is a Morse function whose critical points \( b_i \) have distinct critical values.
- The Morse complex for \( f_x \) is triangular and acyclic, so (after a bit of algebra), there is a pairing \( b_i \leftrightarrow b_{\tau}(i) \) of the critical points such that (after a basis change)
  \[ \partial b_i = b_{\tau}(i) \].
- This pairing locally determines the ruling paths, and restrictions on how these pairings can evolve give rise to the normality condition (via [Barranikov]).
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[Incorporating Barranikov’s condition for normality.]
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- This pairing locally determines the ruling paths, and restrictions on how these pairings can evolve give rise to the normality condition (via [Barranikov]).
Consider the **difference function** [Traynor, Jordan-Traynor, Fuchs-Rutherford]:

\[
\Delta(x, e, \tilde{e}) = f(x, e) - f(x, \tilde{e})
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Note that critical points with positive critical value are in 1-to-1 correspondence with crossings of the \(xy\) diagram.
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Note that critical points with positive critical value are in 1-to-1 correspondence with crossings of the $xy$ diagram.

**Definition**

Given $\eta > 0$ such that $\Delta$ has no critical values in $(0, \eta)$,

\[
GH_\star(f) = H_{\star+N+1}(\Delta \geq \eta, \Delta = \eta; \mathbb{Z}_2)
\]
The set of generating family homologies (taken over all possible generating families) is a Legendrian invariant.
Theorem (Jordan-Traynor, . . .)

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Theorem (Fuchs-Rutherford)

If $f$ generates $K$ and is generic and linear at infinity, then there exists a graded augmentation $\epsilon$ of $(A_K, \partial)$ such that

$$GH_\ast(f) = LCH^\epsilon_\ast(K)$$
An infinite-dimensional analogue of the difference function for generating families is the **action functional** on the space of paths in $\mathbb{R}^3$ beginning and ending on $K$:

$$A(\gamma) = \int_{\gamma} \alpha$$
The Action Functional

An infinite-dimensional analogue of the difference function for generating families is the action functional on the space of paths in \( \mathbb{R}^3 \) beginning and ending on \( K \):

\[
A(\gamma) = \int_\gamma \alpha
\]

Instead of computing relative homology as for \( GH_*(f) \), take motivation from the Morse-Witten-Floer theory of \( A \ldots \)
The Reeb field $R$ satisfies:
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The symplectization of $(\mathbb{R}^3, \alpha)$ is the symplectic manifold

$$(\mathbb{R}^3 \times \mathbb{R}, d(e^t \alpha))$$

$L \times \mathbb{R}$ is a Lagrangian submanifold.
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There is an almost complex structure $J$ for which $\xi$ is complex and that pairs the Reeb and symplectization directions.
Morse-Witten theory:
- Chain complex generated by critical points
- Differential from rigid flowlines between critical points

Legendrian CH:
- Associative algebra generated by Reeb chords
- Differential from rigid $J$-holomorphic $\partial$-punctured disks with $\partial$ on $L \times \mathbb{R}$
A Morse Theory – LCH Dictionary

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Transition to Combinatorics in \((\mathbb{R}^3, \xi_0)\)

For knots in \((\mathbb{R}^3, \xi_0)\), we want to compute Legendrian contact homology combinatorially using \(xy\) diagrams. What happens when we project \(\mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^2_{xy}\)?
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- Rigid \(J\)-holomorphic disks project to rigid holomorphic disks in \(\mathbb{R}^2 = \mathbb{C}\) (Can think of these as smoothly immersed disks by Riemann Mapping Theorem)
Where Are We?

1 Classical Invariants

2 Combinatorial Definitions of Invariants
   - Normal Rulings
   - The Chekanov-Eliashberg DGA
   - Computable Invariants from the DGA

3 Geometry of the Invariants
   - Generating Families
   - Legendrian Contact Homology

4 Extensions
Replacing $L \times \mathbb{R}$ with an exact Lagrangian cobordism $\mathcal{L}$ between $L$ and $L'$ allows us to define a map:

$$\phi_{\mathcal{L}} : LCH(L) \rightarrow LCH(L')$$

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[Ekholm-Honda-Kálmán]

For $L' = \emptyset$, $\phi_{\mathcal{L}}$ is an augmentation, and there are close connections to rulings here.
Why not consider disks in LCH with multiple positive punctures?
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Difficult (but possible) to define an algebraic structure that takes new “bubbling” phenomena into account [Ng]
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This might lead to holomorphic curve-based invariants of transverse knots!
• Geometric ideas lead to an analytic definition of LCH for higher-dimensional Legendrians [Ekholm-Etnyre-Sullivan]; a version of duality still holds [Ekholm-Etnyre-JMS]
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Higher Dimensions

- Geometric ideas lead to an analytic definition of LCH for higher-dimensional Legendrians [Ekholm-Etnyre-Sullivan]; a version of duality still holds [Ekholm-Etnyre-JMS]
- It is possible to use LCH of Legendrian tori to get invariants of smooth knots
- What about rulings? [Henry] Or GF homology?
Lunchtime!