0. Introduction

Part 1 of the documentation contains examples illustrating the basic commands. Part 2 (this notebook) contains more complicated examples and several "case studies" illustrating applications to various problems in combinatorics.

The package should be run first, e.g., by entering

```mathematica
<< posets300.m
```
or by opening the document and clicking on the "run package" button.

6. Sandpile Posets

The "sandpile posets" are a class of posets related to both chip-firing posets and majorization posets (\texttt{MajP[n]}). Essentially, the covering relation is defined as in the majorization lattice, except that only adjacent elements may be transformed. The commands defining these posets are built into the package, but are included here as an illustration.

```mathematica
SOrder[w_] := Module[{outlist = {}, i},
  If[w[-1]] >= 2, AppendTo[outlist, Join[Drop[w, -1], Join[{w[-1] - 1}, {1}]]];
  Do[
    If[w[i]] - w[i+1]] > 1 && w[i]] > 0, AppendTo[outlist, Adjust[i, w]]],
  {i, Length[w] - 1}];
  outlist];

Adjust[j_, w_] := Module[{t = w},
  t[[j]] = w[[j]] - 1; t[[j+1]] = w[[j+1]] + 1;
  t];

SPM[n_] := {SOrder, {{n}}, n^4};
```
Technically, the sandpile poset is the dual of what we have just constructed.
The sandpile posets are always lattices, but (unfortunately) not always distributive.

```
LatticeQ[spm9]
DistributiveLatticeQ[spm9]
True
True
```

Is it an interval in the majorization order?

```
LatticeQ[dspm10]
DistributiveLatticeQ[dspm10]
True
False
```

The answer appears to be "no". Let’s look at the situation a little more carefully. Output from the `Diagram` command
The answer appears to be "no."
Let's look at the situation a little more carefully.

Output from the Diag command is not particularly useful, but it is possible to see much more using the Manipulate option. (Try it yourself, shrinking the dots, shifting, stretching, zooming, etc.)

```
ChangeLabel[majp10, Compact, 1]
Diagram[majp10, spindices, ShowLabels -> 1]
Diagram[majp10, spindices, Manipulate -> True]
```
Here is a "blown-up" version of the diagram, created using the Diagram/Manipulate command.

<table>
<thead>
<tr>
<th>Shifts and Stretches</th>
<th>Special Points</th>
<th>Points and Lines</th>
<th>Label Commands</th>
<th>Save and Reset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtract</td>
<td>Add</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scale</td>
<td>Reflect</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rotate</td>
<td>Translate</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Shifts and Stretches**
- **Stretch**
- **Shift**
- **Zoom**

**Controls**
- **Reset**
- **Save**
- **Load**
Here is a "blown-up" version of the diagram, created using the Diagram/Manipulate command.

Diagram[majp10, spindices, ShowLabels \to 1]

If you explore the diagram carefully you will see that the sandpile lattice is not an interval: the partition (6, 2, 1, 1) is in the sandpile poset, i.e. it can be obtained from (10) by sandpile operations. On the other hand, (7, 1, 1, 1) lies above it in the majorization order, but does not lie in the sandpile poset.

7. Fibonacci Lattices

The r-Fibonacci lattice, denoted by Fib(r), is defined as follows. Let A(r) = \{0,1,...,r\}, and let A(r)* denote the set of all finite words formed from letters in A(r). Define Fib(r) by taking A(r)* as the underlying set, and letting v cover u if u can be obtained from v by changing a 0 in v to an x\neq 0 or by removing the last digit in v if it is non-zero. It is clear that Fib(r) is a graded poset with a unique minimal element. Furthermore, when r = 1, the number of elements at rank n is the Fibonacci number F_{n+1} (where F_1=F_2=1, F_{n+1}=F_n+F_{n-1}).

The first step is to define the covering function for Fib(r). Here fiblattice[r,h] returns a triple \{fibcover, {}, h\}, where fibcover is the covering function (depending on r), {} is the unique minimal element (the empty word), and h is the the rank up to which we wish to build Fib(r). Thus, applying Build to fiblattice[r,h] yields Fib(r) "up to" rank h.

```
fiblattice[r_, h_] := Block[{}, fibcover[w_] := Block[ {coverlist = {}, len = Length[w]}, Do[AppendTo[coverlist, Flatten[{w, i}]], {i, r}]; Do[ If[! (w[[i]] == 0), AppendTo[coverlist, Join[Take[w, i - 1], {0}, Take[w, - (len - i)]]]], {i, len}]; coverlist]; Return[{fibcover, {}, h}];
```
The poset we have constructed is obviously not a lattice, because we have constructed only finitely many ranks. However, the principal order ideals in $\text{Fib}[r]$ are lattices. In fact when $r=1$ they are distributive lattices.
indices = IntervalP[fib105, {}, {1, 0, 1, 1}]
BuildSubPoset[fib105, indices, ideal105]
Diagram[ideal105, ShowLabels -> 1]
{1, 2, 3, 5, 7, 8, 10, 15}
Building Subposet ideal105 ...
Building poset ideal105 ...
Done

DistributiveLatticeQ[ideal105]
True

Next we construct Fib[2].
Build[fiblattice[2, 3], fib203]
ChangeLabel[fib203, Compact, 1]
Diagram[fib203, ShowLabels -> 1]

Building poset fib203 ...

Done

Build[fiblattice[2, 5], fib205]
Diagram[fib205, Manipulate -> True];

Building poset fib205 ...

Done

RGF[fib205]

1 + 2 q + 5 q^2 + 12 q^3 + 29 q^4 + 70 q^5

MaximalChainsDown[fib205]

{1, 1, 1, 1, 2, 1, 1, 1, 4, 2, 1, 1, 4, 1, 1, 2, 1, 1, 1, 6, 4, 2, 1, 1, 6, 4, 12, 1, 1, 6, 2, 1, 1, 6, 1, 4, 2, 1, 1, 4, 1, 1, 2, 1, 1, 1, 1, 8, 6, 4, 2, 1, 1, 1, 8, 6, 4, 32, 16, 12, 1, 1, 8, 6, 2, 1, 1, 8, 6, 32, 1, 1, 8, 4, 2, 1, 1, 8, 4, 16, 1, 1, 8, 2, 1, 1, 8, 1, 1, 6, 4, 2, 1, 1, 6, 4, 12, 1, 1, 6, 2, 1, 1, 6, 1, 1, 4, 2, 1, 1, 4, 1, 1, 2, 1, 1, 1, 1, 8} 

Exercise: what is a formula for the number of maximal chains down from each element of Fib[2]?
indices = IntervalP[fib205, {}, {2, 0, 0}]
BuildSubPoset[fib205, indices, ideal205]
ChangeLabel[ideal205, Compact, 1]
Diagram[ideal205, ShowLabels -> 1]
{1, 3, 7, 8, 16, 17, 18, 19, 20, 40, 41, 44, 47, 100}
Building Subposet ideal205 ... 
Building poset ideal205 ... 
Done

LatticeQ[ideal205]
DistributiveLatticeQ[ideal205]
True
False
For \( r > 1 \) the Fibonacci lattice \( F[r] \) is not distributive.

8. The "Poset Conjecture"

Recall that \( \Omega_{GF}[P,q] \) is the rational generating function for the order polynomial of \( P \).

**Conjecture:** For any poset \( P \), the numerator of \( \Omega_{GF}[p,q] \) has only real roots, and they are all nonpositive.

This was known as the Neggers Conjecture (after Neggers [1978]), or sometimes just the Poset Conjecture, until it was finally disproved in 2007 by John Stembridge.

We will illustrate the calculation on three random posets with 8 elements. Before Stembridge's counterexample, we might have considered this positive evidence for the conjecture.
Build[RandomP[8, .5], testposet];
Diagram[testposet]
num = Numerator[OmegaGF[testposet, q]];
NRoots[num == 0, q]

Building poset testposet ...

Done

q = -6.60276 || q = -1 || q = -0.80293 || q = -0.0943119 || q = 0.
Stembridge showed that the smallest counterexamples to the Poset Conjecture have 17 elements. We conclude this section by constructing one of them.

\[ \text{stex} = \{\{1, 3\}, \{3, 5\}, \{5, 7\}, \{7, 9\}, \{9, 11\}, \{11, 13\}, \{13, 15\}, \{15, 17\}, \{2, 4\}, \{4, 6\}, \{6, 8\}, \{8, 10\}, \{10, 12\}, \{12, 14\}, \{14, 16\}, \{1, 12\}, \{3, 14\}, \{5, 16\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}, \{12, 13\}, \{14, 17\}\}; \]
The "Distributive Lattice Conjecture"

Let $P$ be a poset with 0 and 1. Let $c_i(P)$ be the number of chains of length $i$ from 0 to 1 in $P$ (so, in particular, $c_0(P)=0$ and $c_1(P)=1$). Define the chain polynomial of $P$ by

$$C(P;q) = \sum_{i \geq 1} c_i(P) q^i.$$  

**Conjecture:** Let $L$ be a distributive lattice. Then the polynomial $C(L;q)$ has only real zeros.

This has come to be known as the Distributive Lattice Conjecture. As a matter of fact, it is equivalent to the "Poset Conjecture" described in the previous section. The reason is that $L = J(P)$ for some poset $P$, and it turns out that $C(P;q)$ is related to $\OmegaGF(P;q)$ by a simple change of variables. We will illustrate this by computing both expressions for some randomly generated posets.

We can calculate $C(P;q)$ using the "ChainsBetweenGF" command:
Build[RandomP[5, .3], rand5]
Diagram[rand5]
Build[JoFP[rand5], jprand5]
Diagram[jprand5]
chainpoly = ChainsBetweenGF[jprand5, 1, Card[jprand5]]

Building poset rand5 ...

Done

Building poset jprand5 ...

Done

q + 14 q^2 + 48 q^3 + 60 q^4 + 25 q^5
nn = Numerator[OmegaGF[rand5, q]]
q + 10 q^2 + 12 q^3 + 2 q^4
\[ \text{Expand}\left[ \text{Together}\left[ \frac{\text{nn} \cdot q \cdot t}{1 + t} \right] (1 + t)^5 \right] \]
\[ t + 14 t^2 + 60 t^4 + 25 t^5 \]
\[ \text{NRoots}[\text{chainpoly} == 0, q] \]
\[ q = -1, q = -0.834017, q = -0.462222, q = -0.103761, q = 0. \]
\[ \text{Build}[\text{RandomP}[7, .5], \text{rand7}] \]
\[ \text{Diagram}[\text{rand7}] \]
\[ \text{Build}[\text{JofP}[\text{rand7}], \text{jprand7}] \]
\[ \text{chainpoly} = \text{ChainsBetweenGF}[\text{jprand7}, 1, \text{Card}[\text{jprand7}]] \]
\[ \text{Building poset rand7} \ldots \]
\[ \text{Done} \]

\[ q + 14 q^2 + 66 q^3 + 146 q^4 + 167 q^5 + 96 q^6 + 22 q^7 \]
Diagram[jprand7]

NRoots[chainpoly == 0, q]

q = -1. || q = -1. || q = -1. || q = -0.823497 || q = -0.403263 || q = -0.136876 || q = 0.

Again Stembridge's counterexamples show that the Distributive Lattice Conjecture is false in general:

Build[JofP[stem, jpstem]
chainpoly = ChainsBetweenGF[jpstem, 1, Card[jpstem]]

Building poset jpstem ...

Done

q + 48 q^2 + 936 q^3 + 10 044 q^4 + 67 950 q^5 + 313 466 q^6 +
 1 035 868 q^7 + 2 530 748 q^8 + 4 661 145 q^9 + 6 538 420 q^10 + 6 999 852 q^11 +
 5 683 476 q^{12} + 3 442 872 q^{13} + 1 508 466 q^{14} + 451 880 q^{15} + 82 852 q^{16} + 7016 q^{17}

NRoots[chainpoly == 0, q]

q = -1.06658 || q = -1.06252 || q = -1.03417 || q = -1.02636 ||
q = -0.983865 || q = -0.983659 || q = -0.937142 || q = -0.928866 || q = -0.912186 ||
q = -0.842022 || q = -0.651166 - 0.0182745 i || q = -0.651165 + 0.0182753 i ||
q = -0.359691 || q = -0.196297 || q = -0.100251 || q = -0.0593818 || q = 0.

10. Case Study: Zig-Zag Posets

Next we will perform an extended study of properties of a family of posets called (for obvious reasons) "zigzag posets". Some of the observations made here were discovered using this package.
Build[ZigZag[6], zz6]
Diagram[zz6, ShowLabels -> {1, 2}]
Building poset zz6 ...
Done

Note that the red labels (on the left) are the elements of $P[zz6]$ and the blue labels (on the right) are the indices of those elements. First we construct the lattice of order ideals of $P[zz6]$ and compute its rank generating function.

Build[JofP[zz6], jofzz6]
Diagram[jofzz6]
Building poset jofzz6 ...
Done

$\text{RGF}[\text{jofzz6}]$

$1 + 3q + 4q^2 + 5q^3 + 4q^4 + 3q^5 + q^6$
A study of the roots of these polynomials led us to conjecture and prove (with Michelle Thompson) an explicit formula for the rank generating function. It can be expressed neatly in terms of Chebyshev polynomials:

\[
\operatorname{Expand}\left[ q^3 \operatorname{ChebyshevU}\left[ 3, \frac{1}{2} \left( q + \frac{1}{q} \right) \right] \right]
\]

\[1 + 3q + 4q^2 + 5q^3 + 4q^4 + 3q^5 + q^6\]

Next we compute the linear extensions and topological sortings of \( P[\text{zz5}] \).

```
Build[ZigZag[5], zz5]
Diagram[zz5, ShowLabels -> {1, 2}]
```

Building poset zz5 ...

Done

```
LinearExtensions[zz5]
TopSortings[zz5]
```

\[
\{\{4, 2, 1, 5, 3\}, \{3, 2, 1, 5, 4\}, \{3, 2, 1, 4, 5\}, \{4, 1, 2, 5, 3\}, \\
\{3, 1, 2, 5, 4\}, \{3, 1, 2, 4, 5\}, \{2, 1, 4, 3, 5\}, \{2, 1, 3, 5, 4\}, \\
\{2, 1, 3, 4, 5\}, \{2, 3, 1, 4, 5\}, \{2, 3, 1, 4, 5\}, \{1, 3, 2, 5, 4\}, \\
\{1, 3, 2, 4, 5\}, \{1, 2, 3, 5, 4\}, \{1, 2, 3, 4, 5\}\}
\]

\[
\{\{3, 2, 5, 1, 4\}, \{3, 2, 1, 5, 4\}, \{3, 2, 1, 4, 5\}, \{2, 3, 5, 1, 4\}, \\
\{2, 3, 1, 5, 4\}, \{2, 3, 1, 4, 5\}, \{2, 1, 4, 3, 5\}, \{2, 1, 3, 5, 4\}, \\
\{2, 1, 3, 4, 5\}, \{3, 1, 2, 5, 4\}, \{3, 1, 2, 4, 5\}, \{1, 3, 2, 5, 4\}, \\
\{1, 3, 2, 4, 5\}, \{1, 2, 3, 5, 4\}, \{1, 2, 3, 4, 5\}\}
\]

These objects are essentially equivalent to alternating permutations, that is, permutations having the pattern \( a < b > c < d > e \ldots \), which are of great combinatorial interest. One way to see this is to observe that, if in each topological sorting, we map indices onto the original labels, we obtain a permutation in which the entries 1, 2, \ldots have a “right-left-right-left ...” pattern, and the inverses of such permutations are alternating in the usual sense.
tsort = TopSortings[zz5] /. {1 -> 1, 2 -> 3, 3 -> 5, 4 -> 2, 5 -> 4}
Map[InversePermutation, tsort]

{{5, 3, 4, 1, 2}, {5, 3, 1, 4, 2}, {5, 3, 1, 2, 4}, {5, 3, 4, 1, 2},
 {3, 5, 1, 4, 2}, {3, 5, 1, 2, 4}, {3, 1, 2, 5, 4}, {3, 1, 5, 4, 2},
 {3, 1, 5, 2, 4}, {5, 1, 3, 4, 2}, {5, 1, 3, 2, 4}, {5, 1, 3, 4, 2},
 {1, 5, 3, 2, 4}, {1, 3, 2, 5, 4}, {1, 3, 5, 4, 2}, {1, 3, 5, 2, 4}}

{{4, 5, 2, 3, 1}, {3, 5, 2, 4, 1}, {3, 4, 2, 5, 1}, {4, 5, 1, 3, 2},
 {3, 5, 1, 4, 2}, {3, 4, 1, 5, 2}, {2, 3, 1, 5, 4}, {2, 5, 1, 4, 3},
 {2, 4, 1, 5, 3}, {2, 5, 3, 4, 1}, {2, 4, 3, 5, 1}, {1, 5, 3, 4, 2},
 {1, 4, 3, 5, 2}, {1, 3, 2, 5, 4}, {1, 5, 2, 4, 3}, {1, 4, 2, 5, 3}}

A famous result of D. Andre states that alternating permutations are enumerated by the Eulerian numbers. These are the number whose exponential generating function is Tan[x] + Sec[x]. Thus for example, for n=5, we have

Length[LinearExponential[zz5]]

16

δ[xx, 5] (Tan[x] + Sec[x]) /.x -> 0

16

The subposet of WeakS[5] consisting of the topological sortings of P[zz5] has some interesting properties. In order to display it, we use the command LocateSet to find the indices of corresponding elements in WeakS[5].

Build[WeakS[5], weak5]
indices = LocateSet[weak5, tsort]
Diagram[weak5, indices]

Building poset weak5 ...

Done

{11, 22, 25, 41, 44, 45, 63, 64, 67, 68, 85, 86, 89, 102, 103, 114}

This subposet is actually an interval in P[weak5], i.e. it consists of all elements lying between two fixed elements. This property is somewhat difficult to verify visually, and the following test can be used to confirm it:

IntervalQ[weak5, indices]
indices == IntervalP[weak5, P[weak5][11], P[weak5][114]]

True

True
The observation just made follows from a more general result due to Björner and Wachs ("Permutation statistics and linear extensions of posets", Jour. Combinatorial Theory A 57 (1991)).

Next we extract the sub-poset and take a good look at it.

Next we compute number of maximal chains from the bottom element to each element of P.

Is it a coincidence that the number (16) of maximal chains from bottom to top in the interval of topological sortings of $P[\text{ZigZag}[5]]$ is equal to the total number of such topological sortings (16)? Yes (unfortunately) this is a coincidence, and is not true in general. However, another similar numerical observation turns out not to be a coincidence.

We compute the number of maximal chains from bottom to top in $P[\text{weaks4}]$, a poset with very different structure. This number also equals 16 -- and this is a theorem.
The number of maximal chains in the interval of topological sortings of $P[ZigZag[n]]$ in the weak order of $S_n$ is equal to the number of maximal chains in $S_{n-1}$.

Another illustration (with $n=6$) of this phenomenon:
We defined

Building

Done

Building poset zz6 ...

Done

Building Subposet interval6 ...

Done

Building poset interval6 ...

Done

768

Building poset weak5 ...

Done

768

11. Case Study: 2-Rowed Standard Young Tableaux

This example illustrates an interesting poset whose elements correspond to 2-rowed standard tableaux. The ordering is defined by reading each tableau from left to right, top to bottom (the standard reading order), and considering these permutations as a subposet of the right weak order.

We illustrate this construction with tableaux of shape $\lambda = \{4,3\}$.
Next, we'll illustrate some complicated manipulation of labels. At this point \texttt{P[poset43]} contains a list the permutations that were used to define the sub-order of \texttt{RWeakS[7]}, and these are also mirrored in \texttt{Labels[poset43][[1]]}.

\texttt{P[poset43]}

\{\{1, 2, 3, 4, 5, 6, 7\}, \{1, 2, 3, 5, 4, 6, 7\}, \{1, 2, 4, 5, 3, 6, 7\}, \{1, 2, 3, 6, 4, 5, 7\}, \{1, 2, 3, 7, 4, 5, 6\}, \{1, 2, 4, 5, 3, 6, 7\}, \{1, 2, 4, 6, 3, 5, 7\}, \{1, 2, 3, 7, 4, 5, 6\}, \{1, 2, 5, 7, 3, 4, 6\}, \{1, 3, 4, 5, 2, 6, 7\}, \{1, 3, 4, 6, 2, 5, 7\}, \{1, 3, 4, 7, 2, 5, 6\}, \{1, 3, 5, 6, 2, 4, 7\}, \{1, 3, 5, 7, 2, 4, 6\}\}

It will be convenient to access these in their original tableau form, so we will redefine \texttt{Labels[poset43][[1]]} as tableaux.
ChangeLabel[poset43, Tabify[#, {4, 3}] &], 1]
Labels[poset43][[1]]

\{
\{\{1, 2, 3, 4\}, \{5, 6, 7\}\}, \{\{1, 2, 3, 5\}, \{4, 6, 7\}\},
\{\{1, 2, 4, 5\}, \{3, 6, 7\}\}, \{\{1, 2, 3, 6\}, \{4, 5, 7\}\}, \{\{1, 3, 4, 5\}, \{2, 6, 7\}\},
\{\{1, 2, 4, 6\}, \{3, 5, 7\}\}, \{\{1, 2, 3, 7\}, \{4, 5, 6\}\}, \{\{1, 3, 4, 6\}, \{2, 5, 7\}\},
\{\{1, 2, 5, 6\}, \{3, 4, 7\}\}, \{\{1, 2, 4, 7\}, \{3, 5, 6\}\}, \{\{1, 3, 4, 7\}, \{2, 4, 5\}\},
\{\{1, 3, 4, 6\}, \{2, 5, 6\}\}, \{\{1, 2, 5, 7\}, \{3, 4, 5\}\}, \{\{1, 3, 5, 7\}, \{2, 4, 6\}\}  
\}

We would also like to display these tableaux in compact form, so we'll redefine Labels[poset43][[2]] as actual tableaux.

ChangeLabel[poset43, P[poset43], 2]
ChangeLabel[poset43, Tabify[#, {4, 3}] &], 2]
ChangeLabel[poset43, Grid, 2]
Labels[poset43][[2]]

\{1 2 3 4 1 2 3 5 1 2 4 5 1 2 3 6 1 3 4 5 1 2 4 6 1 2 3 7
5 6 7 4 6 7 3 6 7 4 5 7 2 6 7 3 5 7 4 5 6
1 3 4 6 1 2 5 6 1 2 4 7 1 3 5 6 1 3 4 7 1 2 5 7 1 3 5 7
2 5 7 3 4 7 3 5 6 2 4 7 2 5 6 3 4 6 2 4 6  \}

Diagram[poset43, Manipulate \[ True, Background2 \[ Yellow]
Next we'll separate the tableaux into two classes, according to whether the largest number is in the first row or the second, and display those in the first class as "special" points.

\[
\text{tabs} = \text{Labels[poset43][[1]]} \\
\text{firstrow} = \text{Select[tabs, MemberQ[#[[1]], 7] &]} \\
\text{firstpos} = \text{Table[Position[tabs, firstrow[[j]]], \{j, Length[firstrow]\}]} // \text{Flatten}
\]

\[
\{\{1, 2, 3, 4\}, \{5, 6, 7\}\}, \{\{1, 2, 3, 5\}, \{4, 6, 7\}\}, \\
\{\{1, 2, 4, 5\}, \{3, 6, 7\\}, \{\{1, 2, 3, 6\}, \{4, 5, 7\}\}, \{\{1, 3, 4, 5\}, \{2, 6, 7\}\}, \\
\{\{1, 2, 3, 6\}, \{4, 5, 7\\}, \{\{1, 2, 3, 7\}, \{4, 5, 6\}\}, \{\{1, 3, 4, 6\}, \{2, 5, 7\}\}, \\
\{\{1, 2, 5, 6\}, \{3, 4, 7\\}, \{\{1, 2, 4, 7\}, \{3, 5, 6\}\}, \{\{1, 3, 4, 7\}, \{2, 4, 7\}\}, \\
\{\{1, 3, 4, 7\}, \{2, 4, 7\\}, \{\{1, 2, 5, 6\}, \{3, 4, 6\}\}, \{\{1, 3, 5, 7\}, \{2, 4, 6\}\}}
\]

\[
\{\{1, 2, 3, 7\}, \{4, 5, 6\}\}, \{\{1, 2, 4, 7\}, \{3, 5, 6\}\}, \\
\{\{1, 3, 4, 7\}, \{2, 5, 6\}\}, \{\{1, 2, 5, 7\}, \{3, 4, 6\}\}, \{\{1, 3, 5, 7\}, \{2, 4, 6\}\}
\]

\[
\{7, 10, 12, 13, 14\}
\]

\[
\text{Diagram[poset43, firstpos, Manipulate \to True]}
\]
The selected elements form an interval subposet:

\[ \text{IntervalQ[poset43, firstpos]} \]

True

It is interesting that 2-row tableaux posets are always distributive lattices.

\[ \text{DistributiveLatticeQ[poset43]} \]

True

Also, the ordering on tableaux is identical to that induced by the strong order on permutations.

\[ \text{Build[StrongS[7], ss7]} \]
\[ \text{ind43s = LocateSet[ss7, perms43]} \]
\[ \text{BuildSubPoset[ss7, ind43s, sposet43]} \]

Building poset ss7 ... 

Done

\{1, 5, 20, 23, 54, 66, 71, 144, 159, 164, 306, 311, 330, 569\}

Building Subposet sposet43 ...

Building poset sposet43 ...

Done
Neither of the last two stated properties is true for posets constructed using tableaux with more than two rows.

Example:

```math
\text{posets300-doc2.nb}
```
```
Build[RWeakS[6], ws6]
ind321 = LocateSet[ws6, perms321];
BuildSubPoset[ws6, ind321, poset321]
Diagram[poset321]

Building poset ws6 ...
Done

Building Subposet poset321 ...
Building poset poset321 ...
Done

DistributiveLatticeQ[poset321]
False
```
12. "Rowmotion"

We illustrate an interesting map on the ideals of a poset that has been studied by various people (Brouwer-Schriver, Cameron-Fon Der Flass, Panyushev, Armstrong-Stump-Thomas, Striker-Williams, Roby-Propp).

One defines a bijective map $T: J[P] \rightarrow J[P]$ on ideals of $P$, as follows: if $I$ is an ideal, then $T[I]$ is the ideal generated by the minimal elements of the complement of $P$.

The orbit structure of $T$ has proved to be of great interest. For example, Propp and Roby have shown that, if $P$ is a product of chains, then the average size of an ideal is independent of the orbit. We illustrate their result with the poset $\text{Chain}[3] \circlearrowright \text{Chain}[4]$.

$$T[name_, \text{ideal}_] :=$$

$\text{OrderIdeal}[name, \text{MinElements}[name, \text{Complement}[\text{Range}[\text{Card}[name]], \text{ideal}]]]$

$\text{Build}[\text{CP}[\text{Chain}[3], \text{Chain}[4]], \text{cp34}]$

$\text{Diagram}[\text{cp34}, \text{ShowLabels} \rightarrow 2]$
Building poset cp34 ...

Done

T[cp34, {1, 2, 3, 4, 6}]
{1, 2, 3, 4, 5, 7}

orbit[name_, ideal_] := Module[{orb = {ideal}, next},
    next = T[name, ideal];
    While[next != ideal, orb = Append[orb, next]; next = T[name, next]]; 
orb
]

orb12346 = orbit[cp34, {1, 2, 3, 4, 6}]
Apply[Plus, Map[Length, %]] / Length[%]
{{1, 2, 3, 4, 6}, {1, 2, 3, 4, 5, 7}, {1, 2, 3, 4, 5, 6, 8},
 {1, 2, 3, 4, 5, 6, 7, 9}, {1, 2, 3, 4, 5, 8}, {1, 2, 3, 4, 6, 7}, {1, 2, 3, 5}}

6

orb1 = orbit[cp34, {1}]
Apply[Plus, Map[Length, %]] / Length[%]
{{1}, {1, 2, 3}, {1, 2, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 6, 7, 8, 9},
 {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, {}}

6

orb12 = orbit[cp34, {1, 2}]
Apply[Plus, Map[Length, %]] / Length[%]
{{1, 2}, {1, 2, 3, 4}, {1, 2, 3, 4, 5, 6, 7}, {1, 2, 3, 4, 5, 6, 8, 9},
 {1, 2, 3, 4, 5, 6, 7, 8, 9, 11}, {1, 2, 3, 4, 5, 7, 8, 10}, {1, 3, 6}}

6

orb13 = orbit[cp34, {1, 3}]
Apply[Plus, Map[Length, %]] / Length[%]
{{1, 3}, {1, 2, 3, 6}, {1, 2, 3, 4, 5}, {1, 2, 3, 4, 5, 6, 7, 8},
 {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, {1, 2, 3, 4, 5, 6, 8, 9, 11}, {1, 2, 4, 7}}

6
orb124 = orbit[cp34, {1, 2, 4}]
Apply[Plus, Map[Length, %]] / Length[%]
{{1, 3, 5}, {1, 2, 3, 4, 7}, {1, 2, 3, 5, 6}, {1, 2, 3, 4, 5, 6, 9},
{1, 2, 3, 4, 5, 7, 8}, {1, 2, 3, 4, 5, 6, 7, 8, 10}, {1, 2, 3, 5, 6, 9}}

Check to see that we have all of the orbits:
Join[orb1, orb12, orb13, orb124, orb12346] // Sort
% == (OrderIdeals[cp34] // Sort)
{{}, {1}, {1, 2}, {1, 3}, {1, 2, 3}, {1, 2, 4}, {1, 3, 6}, {1, 2, 3, 4}, {1, 2, 3, 5},
{1, 2, 3, 6}, {1, 2, 4, 7}, {1, 2, 3, 4, 5}, {1, 2, 3, 4, 6}, {1, 2, 3, 4, 7}, {1, 2, 3, 5, 6},
{1, 2, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 7}, {1, 2, 3, 4, 5, 8}, {1, 2, 3, 4, 6, 7},
{1, 2, 3, 4, 5, 6, 9}, {1, 2, 3, 4, 5, 6, 7, 8}, {1, 2, 3, 4, 5, 6, 7, 9}, {1, 2, 3, 4, 5, 6, 8, 9},
{1, 2, 3, 4, 5, 7, 8, 10}, {1, 2, 3, 4, 5, 6, 7, 8, 9}, {1, 2, 3, 4, 5, 6, 7, 8, 10},
{1, 2, 3, 4, 5, 6, 8, 9, 11}, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, {1, 2, 3, 4, 5, 6, 7, 8, 9, 11},
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}}
True

Here's a larger example, with animation.
Build[CP[Chain[7], Chain[11]], cp711]
Diagram[cp711, ShowLabels -> 2]
orb = orbit[cp711, OrderIdeal[cp711, {17, 29}]];
Apply[Plus, Map[Length, %]] / Length[%]
Manipulate[Diagram[cp711, MaxElements[cp711, orb[[k]]]], {k, 1, Length[orb], 1}]

Building poset cp711 ...
13. The Complete Repertoire

We conclude by taking a very simple poset \((P = \text{ZigZag}[4])\) and applying almost every command to either \(P\) or \(J(P)\).
Build[ZigZag[4], zz4]
Build[JofP[zz4], jofzz4]
Building poset zz4 ...
Done
Building poset jofzz4 ...
Done

P[jofzz4]
PGraded[jofzz4]

RankedQ[jofzz4]
Rank[jofzz4]
NK[jofzz4]
RGF[jofzz4]
True

{0, 1, 1, 2, 2, 3, 3, 4}
{1, 2, 2, 2, 1}

1 + 2 q + 2 q^2 + 2 q^3 + q^4

ChangeLabel[jofzz4, Compact, 1]
Diagram[jofzz4, ShowLabels -> {1, 2}]

SpecialPoints[jofzz4, MemberQ[#, 1] &]

{4, 6, 8}
points = MaxAntichain[jofzz4]
links = DilworthCover[jofzz4]
Diagram[jofzz4, points, links, ShowLabels -> {1, 2}]
{6, 7}
{{1, 2}, {2, 4}, {3, 5}, {4, 6}, {5, 7}, {7, 8}}

Diagram[jofzz4, points, links, Manipulate -> True];
Fuse[DilworthCover[jofzz4]] (*Combines links into chains *)
{{1, 2, 4, 6}, {3, 5, 7, 8}}
JoinSubLatticeQ[jofzz4, {2, 4, 5, 7}]
MeetSubLatticeQ[jofzz4, {2, 4, 5, 7}]
False
True
indices = JoinSubLattice[jofzz4, {2, 4, 5, 7}]
BuildSubPoset[jofzz4, indices, jsub]
Diagram[jsub, Manipulate -> True];
{2, 4, 5, 6, 7, 8}
Building Subposet jsub ...
Building poset jsub ...
Done
Ups[jofzz4]
Downs[jofzz4]
StrictUps[jofzz4]
StrictDowns[jofzz4]
{{1, 2, 3, 4, 5, 6, 7, 8}, {2, 4, 5, 6, 7, 8}, {3, 5, 6, 7, 8}, {4, 6, 8}, {5, 6, 7, 8}, {6, 8}, {7, 8}, {8}}
{{1}, {1, 2}, {1, 3}, {1, 2, 4}, {1, 2, 3, 5}, {1, 2, 3, 4, 5, 6}, {1, 2, 3, 5, 7}, {1, 2, 3, 4, 5, 6, 7, 8}}
{{2, 3, 4, 5, 6, 7, 8}, {4, 5, 6, 7, 8}, {5, 6, 7, 8}, {6, 8}, {6, 7, 8}, {8}, {8}, {}}}
{{}, {1}, {1}, {1, 2}, {1, 2, 3}, {1, 2, 3, 4, 5}, {1, 2, 3, 5}, {1, 2, 3, 4, 5, 6, 7}}
CoverRelations[jofzz4]
Covers[jofzz4]
UpDegree[jofzz4]
CoCovers[jofzz4]
DownDegree[jofzz4]

\{{1, 2}, \{1, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{4, 6\}, \{5, 6\}, \{5, 7\}, \{6, 8\}, \{7, 8\}\}

\{{2, 3\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{8, \}, \}\}

\{2, 2, 1, 1, 2, 1, 1, 0\}

\{{1, \}, \{1, \}, \{2, \}, \{2, 3\}, \{4, 5\}, \{5, \}, \{6, 7\}\}

\{0, 1, 1, 1, 2, 1, 2\}

OrderIdeals[jofzz4]
MaxElements[jofzz4, # & , %]

\{{1, 2, 3, 4, 5, 6, 7, 8\}, \{1, 2, 3, 4, 5, 6, 7\},
\{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 5, 7\}, \{1, 2, 3, 4, 5\},
\{1, 2, 3, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 1\}, \{1, 2\}, \{1\}, \}\}

\{{8, \}, \{6, 7\}, \{4, 7\}, \{6, 7\}, \{4, 5\}, \{5, 3\}, \{4, 4\}, \{2, 3\}, \{3, \}, \{2, \}, \{1, \}, \}\}

DualOrderIdeals[jofzz4]
MinElements[jofzz4, # & , %]

\{{\}, \{8, \}, \{6, 8\}, \{7, 8\}, \{4, 6, 8\}, \{6, 7, 8\}, \{4, 6, 7, 8\},
\{5, 6, 7, 8\}, \{3, 5, 6, 7, 8\}, \{4, 5, 6, 7, 8\}, \{2, 4, 5, 6, 7, 8\},
\{3, 4, 5, 6, 7, 8\}, \{2, 3, 4, 5, 6, 7, 8\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\}

\{{\}, \{8, \}, \{6, \}, \{7, \}, \{4, \}, \{6, 7\}, \{4, 7\}, \{5, \}, \{3, \}, \{4, 5\}, \{2, \}, \{3, 4\}, \{2, 3\}, \{1\}\}

MatrixForm[ZetaP[jofzz4]]
MatrixForm[Mu[jofzz4]]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

ToRelation[ZetaP[jofzz4]]

\{{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{1, 7\}, \{1, 8\}, \{2, 2\}, \{2, 4\},
\{2, 5\}, \{4, 6\}, \{2, 7\}, \{2, 8\}, \{3, 3\}, \{3, 5\}, \{3, 6\}, \{3, 7\}, \{3, 8\}, \{4, 4\},
\{4, 6\}, \{4, 8\}, \{5, 5\}, \{5, 6\}, \{5, 7\}, \{5, 8\}, \{6, 6\}, \{6, 8\}, \{7, 7\}, \{7, 8\}, \{8, 8\}\}
MaximalChainsUp[jofzz4]
MaximalChainsDown[jofzz4]
LinearExtensions[zz4]
TopSortings[zz4]

\{5, 3, 2, 1, 2, 1, 1, 1\}
\{1, 1, 1, 1, 2, 3, 2, 5\}
\{\{3, 1, 4, 2\}, \{2, 1, 4, 3\}, \{2, 1, 3, 4\}, \{1, 2, 4, 3\}, \{1, 2, 3, 4\}\}
\{\{2, 4, 1, 3\}, \{2, 1, 4, 3\}, \{2, 1, 3, 4\}, \{1, 2, 4, 3\}, \{1, 2, 3, 4\}\}

ChainsBetweenGF[jofzz4, 1, Card[jofzz4]]

q + 6q^3 + 10q^3 + 5q^4
Together[ZetaPoly[jofzz4]]
Together[Omega[zz4, n]]

\frac{1}{24} \left( 2n + 7n^2 + 10n^3 + 5n^4 \right)
\frac{1}{24} \left( 2n + 7n^2 + 10n^3 + 5n^4 \right)

OmegaGF[zz4, q]
OmegaBarGF[zz4, q]

q + 3q^2 + q^3
(1 - q)^5
q^2 + 3q^3 + q^4
(1 - q)^5

G[zz4, q]
GBar[zz4, q]

\frac{1 + q + q^2 + q^3 + q^4}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4)}
\frac{q^2 + q^3 + q^4 + q^5 + q^6}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4)}
LatticeQ[jofzz4]  
MatrixForm[LJoin[jofzz4]]  
MatrixForm[LMeet[jofzz4]]

True  
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 2 & 5 & 4 & 5 & 6 & 7 & 8 \\
3 & 5 & 3 & 6 & 5 & 6 & 7 & 8 \\
4 & 4 & 6 & 4 & 6 & 6 & 8 & 8 \\
5 & 5 & 5 & 6 & 5 & 6 & 7 & 8 \\
6 & 6 & 6 & 6 & 6 & 6 & 8 & 8 \\
7 & 7 & 7 & 8 & 7 & 8 & 7 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 3 & 1 & 3 & 3 & 3 & 3 \\
1 & 2 & 1 & 4 & 2 & 4 & 2 & 4 \\
1 & 2 & 3 & 2 & 5 & 5 & 5 & 5 \\
1 & 2 & 3 & 4 & 5 & 6 & 5 & 6 \\
1 & 2 & 3 & 2 & 5 & 5 & 7 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{pmatrix}
\]

Build[CP[jofzz4, Chain[4]], bigger]  
Diagram[bigger]

Building poset bigger ...  
Done
BuildSubPoset[bigger, JI[bigger], joinirreducibles]
Diagram[joinirreducibles]
Building Subposet joinirreducibles ...
Building poset joinirreducibles ...
Done
Diagram[bigger, MI[bigger], MICoverRelations[bigger]]

MatrixForm[ZetaMI[bigger]]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
BuildDual[\text{jofzz4, jofzz4dual}]
Diagram[\text{jofzz4dual, ShowLabels \rightarrow 1}]
Diagram[\text{jofzz4dual, ShowLabels \rightarrow 2}]

Building poset jofzz4dual ...

Done

IsomorphicQ[\text{jofzz4, jofzz4dual}]

True

Zap[\text{zz4}]
Zap[\text{jofzz4}]

14. Catalog of Small Posets and Lattices

Along with the package itself, we have supplied two more files, allposets1-7.txt and alllattices1-9.txt. These contain
Along with the package itself, we have supplied two more files, allposets1-7.txt and allposets1-9.txt. These contain descriptions of all non-isomorphic posets of size \( \leq 7 \), and all non-isomorphic lattices of size \( \leq 9 \). They were generated using John Stembridge's Maple package for posets, and then translated into Mathematica format.

This data can be accessed as follows. First be sure that the two files are in a directory that Mathematica searches when input is requested (for example, the AddOns/ExtraPackages folder in the PC version). Then, for example, execute

```mathematica
<< allposets1-7.txt;
```

The data is now recorded in a variable called `AllPosets`. For example, `AllPosets[3]` is a list of all posets of size 3. Each entry consists of a list of relations. Thus, to build the \( k \)th poset, one executes the command `Build[AllPosets[3][[k]],3]`.

```mathematica
AllPosets[3]
```

```mathematica
{{}, {{1, 2}}, {{1, 3}, {2, 3}}, {{1, 2}, {2, 3}}, {{1, 2}, {1, 3}}} 
```

```mathematica
For[k = 1, k \leq 5, k++, Build[AllPosets[3][[k]], 3], sample]; Diagram[sample] // Print
```

Building poset sample ...

Done
Building poset sample ...
Done
Building poset sample  ...

Done

The command **NumPosets** assumes that **allposets1-7** has been read, and returns the number of posets of each size. For example,

```
Table[NumPosets[n], {n, 1, 7}]
{1, 2, 5, 16, 63, 318, 2045}
```

Exercise: check this sequence and the next one (lattices of size n, for 1 ≤ n ≤ 9) by looking in Neil Sloane's book.

```
<< alllattices1-9.txt
```

```
Table[NumLattices[n], {n, 1, 9}]
{1, 1, 1, 2, 5, 15, 53, 222, 1078}
```

We can build all the posets of a size n and test them for various properties. Beware that for large N this may consume
a lot of memory. We illustrate by checking all of the posets with 4 elements to be sure there that no two are isomorphic (output shortened to save space).

\textbf{For}[i = 1, \text{k} \leq 16, \text{k}++,
\text{Build}[(\text{AllPosets}[4][i], 4), \text{test}[\text{k}]])

Building poset test[1] ... 
Done
Building poset test[2] ... 
Done
...
Building poset test[15] ... 
Done
Building poset test[16] ... 
Done
For\[j = 1, j \leq 16, j++\], For\[k = j + 1, k \leq 16, k++\], Print["-----------------------"]; Print["Testing posets ", j, ", ", k]; If[IsomorphicQ[test[j], test[k]], Print["**** Posets ", j, ", ", k, " are isomorphic!"]]]

------------------------------- Testing posets 1, 2 Different number of CoverRelations. ------------------------------- Testing posets 1, 3 Different number of CoverRelations. ------------------------------- Testing posets 1, 4 Different number of CoverRelations. ------------------------------- Testing posets 1, 5 Different number of CoverRelations. ------------------------------- Testing posets 6, 9 Different number of CoverRelations. ------------------------------- Testing posets 6, 10 UpDegree distribution is different. ------------------------------- Testing posets 6, 11 UpDegree distribution is different. ------------------------------- Testing posets 9, 16 Different number of CoverRelations. ------------------------------- Testing posets 10, 11 Number of ones in Zeta matrix is different. ------------------------------- Testing posets 10, 12 DownDegree distribution is different. ------------------------------- Testing posets 10, 13 UpDegree distribution is different. ------------------------------- Testing posets 10, 14 UpDegree distribution is different. ------------------------------- Testing posets 10, 15 Different number of CoverRelations. ------------------------------- Testing posets 13, 15 Different number of CoverRelations. ------------------------------- Testing posets 13, 16 UpDegree distribution is different. ------------------------------- Testing posets 14, 15 Different number of CoverRelations. ------------------------------- Testing posets 14, 16 UpDegree distribution is different. ------------------------------- Testing posets 15, 16 Different number of CoverRelations.
### Appendix: List of All Commands

For more information about a particular command, type

```
Information["<command>", LongForm -> False]
```

For example:

```
Information["DownDegree", LongForm -> False]
```

\[
\text{DownDegree}[\text{name}][\text{x}] \text{ is the number of elements covered by } \text{P}[\text{name}][\text{x}].
\]

To see a list of all Poset commands containing a particular string, type

```
ListCommands["string"]
```

or

```
Information["Posets`*string*", LongForm -> True]
```

For example:

```
ListCommands["Chain"]
```

```
\{
\text{Chain} & \text{MaximalChainsDown} \\
\text{ChainsBetweenGF} & \text{MaximalChainsUp} \\
\text{LongestChain} & \text{---}
\}
```

For a complete list of all commands known to Posets, type

```
ListCommands[]
```

<table>
<thead>
<tr>
<th>AllTableaux</th>
<th>MeetSubLattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMAJ</td>
<td>MeetSubLatticeQ</td>
</tr>
<tr>
<td>Antichain</td>
<td>MI</td>
</tr>
<tr>
<td>ASC</td>
<td>MICoverRelations</td>
</tr>
<tr>
<td>Background1</td>
<td>MinElements</td>
</tr>
<tr>
<td>Background2</td>
<td>MSL</td>
</tr>
<tr>
<td>Build</td>
<td>Mu</td>
</tr>
<tr>
<td>BuildDual</td>
<td>NewLabel</td>
</tr>
<tr>
<td>BuildSubPoset</td>
<td>NK</td>
</tr>
<tr>
<td>Card</td>
<td>NonXP</td>
</tr>
<tr>
<td>Chain</td>
<td>NumLattices</td>
</tr>
<tr>
<td>ChainsBetweenGF</td>
<td>NumP</td>
</tr>
<tr>
<td>ChangeLabel</td>
<td>NumPosets</td>
</tr>
<tr>
<td>CharPoly</td>
<td>Omega</td>
</tr>
<tr>
<td>CleanRank</td>
<td>OmegaBar</td>
</tr>
<tr>
<td>CoCovers</td>
<td>OmegaBarGF</td>
</tr>
<tr>
<td>Compact</td>
<td>OmegaGF</td>
</tr>
<tr>
<td>ContractionLattice</td>
<td>OrderIdeal</td>
</tr>
<tr>
<td>CoverRelations</td>
<td>OrderIdealQ</td>
</tr>
<tr>
<td>Covers</td>
<td>OrderIdeals</td>
</tr>
<tr>
<td>CP</td>
<td>OrdinalSum</td>
</tr>
<tr>
<td>CProduct</td>
<td>OS</td>
</tr>
<tr>
<td>DES</td>
<td>P</td>
</tr>
<tr>
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<td>PGraded</td>
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<td>PolyToGF</td>
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<tr>
<td>DistributiveLatticeQ</td>
<td>PosetP</td>
</tr>
<tr>
<td>Div</td>
<td>RandomP</td>
</tr>
<tr>
<td>DownDegree</td>
<td>Rank</td>
</tr>
</tbody>
</table>
For more information about each command, please consult the "usage" section of the package. This gives the precise syntax for each command, and can even be printed out as a compact "user's manual", if desired.