Math 395b: Advanced Topics in Combinatorics  Spring 2014

Course Description: The subtitle of this course is “Strange Phenomena in Combinatorics and Probability”. It will cover a selection of problems whose solutions are surprising or counter-intuitive, and which therefore invite deeper investigation.

Probability is full of intuition-defying results: elementary phenomena which are less (or more) likely then they might seem. In combinatorics, if two seemingly unrelated counting problems have the same answer, then this is also “strange” and suggests a closer look.

The structure of the course will be problem-based, but its methodology will illustrate central themes of enumerative combinatorics. The course will provide an introduction to and overview of that area, including its research frontiers.

Topics covered will include a selection from the following (see the end of this document for more detailed descriptions in several cases).

- Pattern-matching games and the combinatorics of strings.  See (1) below.
- Paradoxes in probability, based on simple counting.  See (2) below.
- Catalan problems: a multitude of simple enumeration problems leading to the same counting sequence $1, 2, 5, 14, 42, 132, 429, \ldots$.
- Partition identities and combinatorial correspondences: easy-to-state (but often difficult) problems of finding bijections between sets of the same cardinality.  See (3) below.
- Enumeration of lattice paths, polyominoes and other problems involving random walks and cell-growth models.
- Problems on random permutations: shuffling and order statistics.
- Tiling problems: rectangles, Aztec diamonds, and other geometric shapes: problems with interesting connections to statistical physics.  See (4) below.

Textbook: There is no official textbook. The course will be based on notes and articles posted on Moodle. In addition, there are two recommended references, one for probability and the other for combinatorics. Both of these excellent books are available for free downloading on the web.


Prerequisites: The stated prerequisite for this course is Math 333 or consent of instructor, but in fact Math 215 should be enough in most cases. We will not use much algebra beyond linear algebra, but some mathematical experience and/or sophistication may be helpful. No specific background in probability or combinatorics will be assumed: elements of these subjects
will be developed as needed. *Perhaps the most important prerequisite is interest in solving challenging problems like the ones described in this handout.*

**Course Schedule:** Three lectures per week (MWF 1:30-2:30).

**Homework, Tests, Grades:** There will be regular homework assignments, two midterm exams (in-class and take-home), and a final (take-home). Dates will be announced. In addition, there will be a term project, and (depending on the size of the class) opportunities to present this material to each other orally.

**Math Question Center:** I encourage collaboration on the homework, and many students find it useful to work together in the Math Question Center (Sunday through Thursday 7-9PM, Hilles 011/012).

**Electronic Resources:** Moodle will be the primary source for course materials. You are responsible for getting the weekly homework assignments from Moodle. I will also post solutions to homework and tests, handouts, links to course-related websites, and other materials that might be quite important (e.g. corrections to and hints to homework assignments, reminders of test dates, etc.).

Mathematica will be used frequently as an aid to computation. You are probably already familiar with Mathematica. If not, this would be a good time to make your acquaintance with it. I will illustrate Mathematica commands and techniques that will be especially useful in this course.

**Collaboration:** I encourage collaboration on the homework. Indeed, I expect you will learn a great deal about this course from each other. It will be to your advantage to form study groups, and many students facilitate this by working together in the Math Question Center.

Collaboration on homework naturally raises the question, "how much is OK?" I expect that you will share ideas, and perhaps work together at a blackboard, but eventually each student must write up his/her work independently, without reference to another student's work or to written work that has been produced jointly. Verbatim copying from another person's paper or blackboard work is definitely "not OK". The safest approach is to write up your final solutions in a different place, and on a fresh sheet of paper.

It's important not to misunderstand these guidelines, so please ask me if you have any questions. You might also want to refer to the department's published guidelines on homework collaboration, which are available on the department website.

It goes without saying that collaboration on tests is never permitted. All inquiries about problems on the tests should be directed to me.

**Honor Code Principle:** *You must never present others' work as your own.* If you have used other students' work in the preparation of homework you must acknowledge it. If you obtain solutions to assigned problems (on homework or take-home tests) from sources other than the textbook or class notes, you must acknowledge such sources. *This especially applies to material obtained electronically, e.g., on the web.*

If there are questions about honor code issues, you should seek clarification and guidance from me.
Sample Topics:

(1) **Pattern matching games.** You and I play a game as follows. We each pick a string of heads and tails (for example, you might pick TTHH and I pick HTHH). A coin is flipped repeatedly, and the person whose string appears first is the winner. Who has the best chance of winning? Or are the chances even? (Answer: the person who picked TTHH wins more than 60% of the time.) Who has to wait longer, on average, before his/her string appears? (Answer: the person who picked TTHH has to wait longer! The average waiting time for TTHH is 20 flips, but the average waiting time for HTHH is 18 flips. This is really strange!). We will study a method for calculating these probabilities and waiting times, and explore other interesting features of the problem.

(2) **Conditional probability paradoxes.** Let $P$ be the probability that a poker hand contains at least two aces, given that it contains at least one ace. Let $Q$ be the probability that a poker hand contains at least two aces, given that it contains the ace of spades. Which is bigger, $P$ or $Q$? Or are they the same? (Answer: they are NOT the same, but I will let you figure out which is bigger.)

(3) **Counting partitions of integers.** Let $f(n)$ equal the number of ways of writing the integer $n$ as a sum of distinct positive integers less than or equal to $n$, and let $g(n)$ equal the number of ways of writing $n$ as a sum of odd integers less than or equal to $n$. For example, $f(5)$ and $g(5)$ are both equal to 3, since

\[
5 = 5 \quad , \quad 5 = 4 + 1 \quad , \quad 5 = 3 + 2
\]

and

\[
5 = 5 \quad , \quad 5 = 3 + 1 + 1 \quad , \quad 5 = 1 + 1 + 1 + 1 + 1
\]

This is no coincidence: we will show that $f(n) = g(n)$ for all values of $n$. Why is this true? Can you establish a direct connection (a “bijection”) between partitions with distinct parts and partitions with odd parts?

There are many problems like this, some involving partitions of integers and some involving other combinatorial objects. In many cases (including this one) an explicit bijection can be constructed; in others the equality of counts remains a relative mystery, i.e., we can prove result is true but there is no simple combinatorial explanation for it.

(4) **Tiling problems.** The **Aztec Diamond of order $N$** is an arrangement of $2N(N+1)$ cells in $2N$ rows and $2N$ columns, in a shape illustrated by the following figure on the left, which is an Aztec Diamond of order 4:

A domino is a pair of two adjacent cells, oriented either horizontally or vertically. A domino tiling of the Aztec Diamond is a covering of all the cells by disjoint dominos. There are lots of domino tilings of an Aztec Diamond of order $N$. In fact, the exact number is

\[
t(N) = 2^{N(N+1)/2}.
\]
The figure on the right above illustrates one of the $2^{10} = 1024$ domino tilings of the Aztec Diamond of order 4. Why is $t(N)$ always equal to a power of 2? Is there a way to associate tilings with binary strings of length $N(N + 1)/2$?

The answer is YES, but it is not at all simple. The solution leads to many other interesting questions about tilings. A brief look at random tilings of the Aztec diamond (chosen with uniform probability $1/T(N)$) reveals some truly amazing phenomena. For example, random tilings tend to “freeze” near the corners in a circular pattern — this has been called the “Arctic Circle” phenomenon. For more information about this problem, see the colorful and informative website: http://faculty.uml.edu/jpropp/tiling/www/.