Limit Equilibria of Uni-dimensional Bargaining Games under Super-Majority Rule

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Abstract

I study equilibria in uni-dimensional bargaining games under $q$-majority rule. The limit equilibria as players become perfectly patient are characterized. The players whose ideal policies are $q^{th}$ from the right and left, denoted $l$ and $r$, respectively, play an important role. The limit policy is either the ideal policy of some player between $l$ and $r$ (not necessarily the median), or is the solution to a bilateral asymmetric Nash Bargaining problem between $l$ and $r$. Which of these outcomes obtains, and the bargaining weights in the asymmetric Nash Bargaining game are determined endogenously by the ideal policies of all agents. The model provides foundations for the endogenous separation of agents with disparate preferences into cohesive voting blocs that are represented by agents with non-median preferences.

Key Words: Bargaining, Super-majority Rules, Endogenous Factions, Core.

JEL Codes: C72, C78, D7

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1 Introduction

In this paper, I analyze bargaining outcomes over a one-dimensional space under the canonical framework of Baron and Ferejohn (1989). Let $X \subset \mathbb{R}$ be a uni-dimensional policy space. There is a committee or legislature consisting of $n$ people that may adopt a policy $x \in X$ by an affirmative vote of $q$ agents, where $q > \frac{n}{2}$ constitutes a (super)-majority. Examples of such situations may include the adoption of a tax rate by a legislature, the choice of a legal rule by the Supreme Court (see Lax & Cameron (2011), Parameswaran, Cameron & Kornhauser (2018)). It is standard in the literature to assume that agents have preferences that are concave and single-peaked over the policy space. I specialize these preferences further by focusing on preferences that satisfy a ‘symmetry’ assumption (see Cardona and Ponsati (2011)), which amounts to asserting that each agent’s preferences are characterized by a common convex loss function centered at the agent’s ideal policy. Absolute value loss and quadratic loss are two examples of such preferences commonly deployed in the applied literature (see ***)

The symmetry assumption implies that, in equilibrium, the coalitions that support and reject any proposal are both connected. This, in turn, implies that any successful proposal must receive the support of both the $q^{th}$ agent from the right and the $q^{th}$ agent from the left (and all the agents between them). I refer to these as the left and right decisive agents, respectively. The symmetry assumption also guarantees the existence of a unique sub-game perfect equilibrium in symmetric strategies (see Cardona and Ponsati (2011)). Predtetchinski (2011) shows that, as the cost of delay becomes costless (i.e. as $\delta \to 1$), equilibrium proposals in uni-dimensional bargaining games converge to a unique limit.$^{1}$

In this paper, I provide a simple and intuitive characterization of these limit equilibria. I show that the limit equilibrium policy will generically coincide with either the asymmetric

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$^{1}$In fact, Predtetchinski (2011) shows more strongly that, absent the symmetry assumption when there are generically multiple equilibria, proposals in all equilibria converge to same limit.
Nash Bargaining solution to the bilateral bargaining problem between the left and right decisive agents, or to the ideal policies of some agent recognized with positive probability whose ideal policy lies strictly between the ideal policies of the left and right decisive agents. Which of these outcomes prevails depends on the recognition probabilities of each agent, and the spatial alignment of their ideal policies. When the limit equilibrium corresponds to the ideal policy of some agent, I say that agent is pivotal. Under simple majority rule with an odd number of players, the median agent is pivotal. (Cho and Duggan (2009) show that this will be true under quite general assumptions.) However, under a super-majority rule, the pivotal agent need not be the median. Indeed, under unanimity, there is some arrangement of recognition probabilities and ideal policies that makes every agent other than the most extreme ones, pivotal. I provide comparative statics on the size of the super-majority rule, and show that larger super-majority requirements generate more moderate outcomes.

There are several reasons that motivate the focus on limit equilibria. First, this limit captures the outcome of bargaining when players can make rapid and costless counter-proposals, which we might believe more accurately captures the dynamics of bargaining. Second, and relatedly, costly delay gives an arbitrary advantage to the who is recognized to make the first proposal; focusing on limit equilibria removes this advantage that stems purely from the stylized bargaining protocol. Third, considering the limit as counter-proposals become cheap allows for more appropriate comparisons between the predictions of bargaining models and other approaches, such as those predicting outcomes in the core. Fourth, as I will show, whilst the bargaining model admits a unique equilibrium whenever delay has some positive cost, there will be a continuum of equilibrium under super-majority rule when delay is costless. Characterizing the limit equilibrium as the cost of delay approaches zero allows the modeler to select a unique focal equilibrium from amongst the many that exist when the cost is truly zero. Finally, limit equilibria are often simpler to characterize than equilibria of

\[\text{2A 'core' outcome is one for which there does not exist another proposal that is strictly preferred by a decisive coalition. Models that predict outcomes in the core implicitly assume that non-core outcomes would be replaced, which assumes counter-proposals are not too expensive to propose.}\]
bargaining games with costly delay. This follows whenever limit equilibria are characterized as the solution to some decision procedure, such as the Nash Bargaining solution, rather than as the fixed point of some mapping induced by a game.

The intuition for the main result is as follows: I have already argued that, in equilibrium, any policy that is accepted must have the support of both the left and right decisive voters. Let $\underline{x}$ be the lowest policy that the right decisive agent will accept, and $\overline{x}$ be the highest policy that the left decisive agent will accept. Then, the equilibrium acceptance set is the interval $[\underline{x}, \overline{x}]$. Given the convergence result in Predtetchinski (2011), it follows that $\lim_{\delta \to 1} \underline{x} = \mu = \lim_{\delta \to 1} \overline{x}$. Let $x_j$ be the ideal policy of player $j$ and conjecture that $\mu \in (x_{i-1}, x_i)$ for some $i$. Then, for $\delta$ large enough, $x_{i-1} < \underline{x} < \overline{x} < x_i$ — and so, in equilibrium, agents $1, \ldots, i-1$ will propose $\underline{x}$ and agents $i, \ldots, n$ will propose $\overline{x}$. It is as if the players separate into two factions, with all members of the same faction making the same equilibrium proposals. Moreover, by construction, $\underline{x}$ and $\overline{x}$ were determined by the preferences of the right and left decisive agents. Hence, we can think of the left decisive agent as bargaining on behalf of the left faction, and the right decisive agent as bargaining on behalf of the right faction, with the bargaining weights being proportional to the recognition probabilities of all of the agents in their respective factions. As $\delta \to 1$, this coincides with the asymmetric Nash bargaining solution between the left and right decisive agents (see Binmore, Rubinstein and Wolinsky (1986)).

Notice that the separation into factions was endogenous to the equilibrium. Hence, for the asymmetric Nash bargaining solution to indeed be equilibrium consistent, it must be that this solution lies in the interval $(x_{i-1}, x_i)$ — so that players separate into the factions as conjectured. I show in the paper that there is a unique player $i^*$ that determines the composition of factions, in equilibrium. There are two possibilities. For a range of recognition probabilities and ideal policy arrangements, the factions $\{1, \ldots, i^*-1\}$ and $\{i^*, \ldots, n\}$ induce a recognition probability weighted Nash Bargaining solution that is equilibrium consistent.
— i.e. which falls in the required interval \((x_{i^*-1}, x_{i^*})\). Outside this range of parameters, the following problem arises: If \(i^*\) is conjectured to be in the left faction, then the location of the induced Nash Bargaining solution will cause \(i^*\) to defect to the right faction, and vice versa. Player \(i^*\) is pivotal. The only possibility is that the limit equilibrium coincides with \(i^*\)’s ideal policy, \(x_{i^*}\).

Comparative statics. [To be written.]

This paper builds on existing work that studies bargaining in a one-dimensional framework. Banks and Duggan (2000), extending Baron and Ferejohn (1989), show, in a framework that admits mine as a special case, that equilibria no delay equilibria exist and that equilibria must be in no-delay when \(\delta < 1\). Cho and Duggan (2003) show that equilibria in a one dimensional framework are unique when preferences are quadratic, but that multiple pay-off variant equilibria may exist for generic concave preferences. Cardona and Ponsati (2011) show that equilibria will be unique if preferences are ‘symmetric’ — i.e. if they are characterized by a common loss function centered at each agent’s ideal policy. Predtetchinski (2011) shows that for generic concave preferences, the limit equilibria will be unique, even if, for \(\delta < 1\), the game admits multiple equilibria. Moreover, Predtetchinski (2011) characterizes this limit equilibrium as the generalized zero of a characteristic function. In the analysis below, I show that combined with the symmetry assumption in Cardona and Ponsati (2011), the zeros of this characteristic function generically correspond to the asymmetric Nash bargaining solutions to a bargaining game between the left and right decisive voters, respectively. Other papers establish the properties of uni-dimensional bargaining under alternative or particular bargaining protocols. Cardona and Ponsatí (2007) study equilibria when the recognition rule follows a deterministic process. Herings and Predtetchinski (2010) study equilibria under unanimity rule when the recognition rule follows a Markov process.

This paper also contributes to a literature that demonstrate the links between outcomes of non-comparative bargaining games to other solution concepts. Binmore, Rubinstein and
Wolinsky (1986) show that the equilibria of the two player alternating-offers bargaining model of Rubinstein (1982) converge to the Nash bargaining solution as players become perfectly patient. Cho and Duggan (2009) show, in a one-dimensional model under simple majority rule with multiple players and random recognition, that the limit equilibrium selects the median voter’s ideal policy. Imai and Salonen (2000) show in a one dimensional framework with players grouped into opposing factions, that equilibria are characterized by a ‘Representative Nash solution’. In this paper, I show that, in the limit equilibrium, it is as if the players separate into endogenous opposing factions, and that the limit equilibrium is the result of Nash bargaining between representatives of these two factions. Hart and Mas-Colell (1996), Laruelle and Valenciano (2008) and Miyakawa (2008) show that, when the space of preferences is comprehensive\(^3\), the limit equilibrium corresponds to the solution of a generalize Nash bargaining problem of the form: \( \Pi_i (u_i(x) - u_i(d))^{p_i} \) — i.e. the solution maximizes the ‘weighted’ product of the agents’ surplus from the bargain, where the weights correspond to recognition probabilities. Laruelle and Valenciano (2008) additionally demonstrate the relationship of this solution to the Shapley-Shubik value (see Shapley and Shubik (1954)). As should be clear, payoffs in a one-dimensional space with concave preferences are not comprehensive. This distinction explains the differences in the results from those papers and mine.

The remainder of this paper is organized as follows: Section 2 outlines the bargaining framework. Section 3 characterizes the limit equilibrium and explores several properties of this limit. Section 4 provides several extensions, and Section 5 concludes.

\(^3\)Let \( u \) be a payoff vector and let \( U \) be the utility possibility set. The payoff space is comprehensive if \( u \in U \) and \( u' \leq u \) implies that \( u' \in U \).
Framework

Let $X \subset \mathbb{R}$ be a compact interval denoting a set of outcomes. There are $n$ agents, $i = 1, \ldots, n$, who bargain over the outcome to be implemented. Each player $i$ has expected utility preferences represented by a continuous, concave expected utility index $u_i(x)$ that achieves a maximum at $x_i$. The players are ordered by their ideal policies, so that $x_1 \leq x_2 \leq \ldots \leq x_n$. Preferences satisfy the symmetry axiom\(^4\) in Cardona and Ponsati (2011). This implies that there exists a continuous, concave function $u$ such that $u_i(x) = u(x - x_i)$ for each $i$. Note that this notion of symmetry is different from the usual notion, that $u_j(x_j + \varepsilon) = u_j(x_j - \varepsilon)$ for every $\varepsilon > 0$. I assume $u(z)$ is continuously differentiable for all $z \neq 0$. (The exception at the ideal policy allows the framework to include absolute value loss functions, which are commonly used in the literature.)

The bargaining protocol is the standard procedure in Baron and Ferejohn (1989) and Banks and Duggan (2000). In each round of bargaining, a player is randomly selected to propose a policy. Let $p_i \geq 0$ be the probability that player $i$ is recognized to make a proposal. A proposal $y$ is accepted if it receives the assent of at least $q > \frac{n}{2}$ agents. In the event of disagreement, the players adjourn and reconvene for another round of bargaining in the following period. This process continues until agreement is reached. Players discount the future at a common rate $\delta \in [0, 1)$. As is common in the literature (see Cardona and Ponsati (2011), Predtetchinski (2011), amongst many others), I assume $u_i(x) \geq 0$ for all $x \in X$ and normalize the disagreement payoff to 0. This implies that every policy is preferred to disagreement. In section 4, I consider an extension in which disagreement entails reversion to a status quo policy $x_{sq} \in X$.

A strategy for player $i$ is a pair $s_i = (y_i, A_i)$, where $y_i \in X$ is the policy proposed whenever $i$ is recognized as the proposer, and $A_i \subset X$ is the set of policies that player $i$ will accept

\(^4\)Formally, for any pair of players $i$ and $j$, and for every $\varepsilon > 0$, $u_i(x_i + \varepsilon) = u_j(x_j + \varepsilon)$, provided that $x_i + \varepsilon, x_j + \varepsilon \in X$. 

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I solve for stationary sub-game perfect equilibria. I limit attention to strategies that satisfy the weak dominance property; each player supports a proposal only if the utility from having the policy implemented is at least as large as the utility from disagreement.

3 Analysis

3.1 Preliminaries

To begin, I briefly summarize several features of the equilibrium that have been previously established. In a framework that nests this model, Banks and Duggan (2000) establish that equilibria must be in no delay, and that no delay equilibria exist. The acceptance set for each player, then, is given by:

\[ A_i = \left\{ y \in X \mid u(y - x_i) \geq \delta \sum_j p_j u(y_j - x_i) \right\} \]

Since \( u \) is continuous and concave, there exist thresholds \( x_i^L < x_i < x_i^R \) s.t. \( A_i = [x_i^L, x_i^R] \). Moreover, given the symmetry assumption, these thresholds respect the ordering of players, so that \( x_1^L \leq \ldots \leq x_n^L \) and \( x_1^R \leq \ldots \leq x_n^R \). (This property was first noted in Cho and Duggan (2003), and appears in Cardona and Ponsatí (2007).)

Let \( C = \{C \subset \{1, \ldots, n\} \mid |C| \geq q\} \) be the set of winning coalitions. Let \( A_C = \cap_{i \in C} A_i \) be the set of proposals that will be accepted by coalition \( C \), and let \( A = \cup_{C \in \mathcal{C}} A_C \) be the social acceptance set. Given that acceptance sets are ordered, every equilibrium coalition must be connected. It follows that every equilibrium coalition must include player \( n - q + 1 \) (the \( q^{th} \) player from the right) and player \( q \) (the \( q^{th} \) player from the left). There is no winning coalition that will support a policy \( y < \overline{x_q} \) or a policy \( y > \overline{x_{n-q+1}} \). The social acceptance
set is \( A = \left[ x_q, x_{n-q+1} \right] \). For notational simplicity, let \( l = n - q + 1 \) and \( r = q \). I refer to \( l \) and \( r \) as the left and right decisive voters, respectively. Given this discussion, we have the following result:

**Proposition 1.** There exists a unique stationary equilibrium. In equilibrium, there is no delay, and strategies are characterized by two thresholds, \( x_r \) and \( x_l \) (with \( x_r < x_l \)) satisfying:

1. When recognized, player \( i \) proposes:

\[
y_i = \begin{cases} 
  x_r & x_i < x_r \\
  x_i & x_i \in [x_r, x_l] \\
  x_l & x_i > x_l
\end{cases}
\]

where \( x_r < x_r \) and \( x_l > x_l \) satisfy:

\[
u(x_r - x_r) = \delta \sum_j p_j u(y_j - x_r)
\]

\[
u(x_l - x_l) = \delta \sum_j p_j u(y_j - x_l)
\]

2. Player \( i \)'s acceptance set is: 

\[
A_i = \left\{ y \in X \mid u(y - x_i) \geq \delta \sum_j p_j u(y_j - x_i) \right\}.
\]

**Proof.** See Cardona and Ponsati (2011)

I provide an intuitive account of this result. Let \( E[y] = \sum_j p_j y_j \) be the expected proposal in the continuation game. Since the equilibrium is in no-delay, the players understand that if the current proposal is not accepted, then equilibrium proposals in continuation game will be accepted. Suppose player \( i \) is recognized as the proposer. Concavity of preferences implies that an offer of \( E[y] \) by player \( i \) will be accepted by all agents. (Continuity and discounting further imply that policies within a neighborhood of \( E[y] \) will be accepted as
well.) Then, starting from $E[y]$, player $i$ has an incentive to pull the offer towards her ideal policy. For concreteness, suppose $x_i < E[y]$. As the proposal moves further from $E[y]$, it becomes less desirable from the perspective of players whose ideal policies were above $E[y]$. The symmetry assumption along with concavity of $u$ implies that player $n$ will be the first player whose support is lost, then player $n-1$, and so on. (This follows because the coalitions that accept and reject proposals are both connected.) Since the proposer only requires the support of $q$ agents, she can continue to pull the proposal towards her ideal until either her ideal policy is reached, or she would lose the support of the right decisive player. Analogously, if $x_i > E[y]$, then the proposer can continue to pull the policy up until she would lose the support of the left decisive player.

In this paper, I focus on the behavior of equilibria as $\delta \to 1$. We can interpret this limit in several ways. On the one hand, this is the equilibrium that obtains as the players become perfectly patient. We can also think of this as capturing bargaining under a dynamic where players can make counter-proposals arbitrarily quickly, or when the cost of making counter-proposals becomes negligible. Similarly, it might represent the case where the risk that negotiations breakdown following disagreement, becomes arbitrarily small. The following result reproduces Theorem 3.6 from Predtetchinski (2011).

**Proposition 2.** There exists $\mu \in X$ s.t. $\lim_{\delta \to 1} x_r = \mu = \lim_{\delta \to 1} x_l$.

An implication of Proposition 2 is that, as $\delta \to 1$, equilibrium proposals converge, and become independent of the identity of the proposer. Predtetchinski (2011) shows that the limit equilibrium is the generalized root of a characteristic equation which I discuss in greater detail, below.

The limit equilibrium of the bargaining game is related to the notion of the core. Given a (super)-majority rule $q$, the core $K$ refers to the set of policies $x$ for which there does not exist a different policy $x'$ that is strictly preferred by at least $q$ players. In the one-dimensional
context, the core is simply the subset of policies between the ideal policies of the left and right
decisive agents; $\mathcal{K}(q) = [x_l, x_r]$. Now, if $n$ is odd and $q = \frac{n+1}{2}$ represents a simple-majority,
then the core is simply the ideal policy of the median player ($\mathcal{K}(q) = \{x_{med}\}$). This is the
median voter result, due to Black (1948). If $n$ is even, or if $q$ represents a super-majority,
then the core is an interval, and so contains a continuum of policies.

Banks and Duggan (2000) show that, if $\delta = 1$, then any outcome in the core can be sustained
as an equilibrium of the bargaining game. Combining this result with the uniqueness result
in Cardona and Ponsati (2011) and the convergence result in Predtetchinski (2011) gives the
following insight: When $\delta < 1$, the bargaining game admits a unique equilibrium, and as
$\delta \to 1$, the equilibrium policies converge to a unique outcome. However, at $\delta = 1$, the set
of equilibrium proposals explodes to include the entire core. The equilibrium is not lower-
hemicontinuous. However, since the equilibrium correspondence is upper-hemicontinuous
(see Banks and Duggan (2000)), the limit policy $\mu$ must be contained in the core. Hence,
we can interpret the limit equilibrium as selecting a particular core policy — it is the robust
core policy that can be sustained as the outcome of a bargaining game when the cost of
delay is small.

### 3.2 The Faction Formation Game

Before characterizing the limit equilibrium of this game, a brief digression. Consider the
following two stage game, which I dub the *faction formation* game. In the first stage, each
player $i$ allocates his support between two faction $L$ and $R$. Let $\rho_i \in [0, 1]$ be the amount of
support given to faction $R$, and $1 - \rho_i$ be the support given to faction $L$. Let $\rho = \sum_i p_i \rho_i$ be
the total support for faction $R$, and $1 - \rho$ be the total support for faction $L$. In the second
stage, a policy $b$ is selected according to asymmetric Nash Bargaining between the left and
right decisive agents, with weights proportional to the total supports for each faction.
It should be clear that the second stage outcome follows mechanically from the first stage choices of each agent. We have:

\[ b_\rho = \arg \max_{b \in X} u(b - x_l)^{1-\rho} u(b - x_r)\rho \]

If the ideal policies of the left and right decisive agents coincide, then the faction formation game is trivial. Any profile of first stage strategies is equilibrium consistent, and the second stage outcome is simply the ideal policy of the decisive agents. The problem becomes more interesting when the decisive agents have different ideal policies. If so, then in the second stage, the chosen policy \( b_\rho \) is monotonically increasing in \( \rho \). (In fact, \( b_\rho \) increases from \( x_l \) to \( x_r \), as \( \rho \) increases from 0 to 1.) Then, for the most part, players will not optimally split their support between the factions, since doing so pulls the resulting policy in opposite directions, negating the overall effect. In fact, in equilibrium, an agent will only split her support between the factions, if the resulting second stage policy coincides with her ideal policy. This implies the following proposition:

**Proposition 3.** There exists a unique policy \( b^* \) chosen in any Nash Equilibrium. Furthermore, if \( x_l < x_r \), then in any equilibrium, \( \rho_i = 0 \) for any player \( i \) with \( x_i < b^* \), and \( \rho_i = 1 \) for any player \( i \) with \( x_i > b^* \).

Proposition 3 shows that the policy chosen in any equilibrium of the faction formation game is unique. Moreover, if \( x_l < x_r \), the equilibrium strategies of all players whose ideal policies do not coincide with the equilibrium policy are also unique. (The intuition for the latter result follows immediately from the preceding paragraph.) There are two potential sources of multiple equilibria. First, if \( x_l = x_r \), then any profile of support levels is equilibrium consistent. Second, if \( x_l < x_r \) and there several players whose ideal policies coincide with the equilibrium policy, then two profiles \((\rho_1, \ldots, \rho_n)\) and \((\rho'_1, \ldots, \rho'_n)\) may be equilibrium consistent provided that \( \sum_{i: x_i = b^*} \rho_i = \sum_{i: x_i = b^*} \rho'_i \). In either case, the induced policy is the same.
Just in case there are multiple equilibria, I apply the following refinements to select a unique equilibrium. I say that a strategy profile of support levels \((\rho_1, \ldots, \rho_n)\) is **monotone** if \(i < j\) implies \(\rho_i \leq \rho_j\). Moreover, I say that a strategy profile \((\rho_1, \ldots, \rho_n)\) is **almost-pure** if \(\rho_i \in (0, 1)\) for at most one player. If a strategy profile is **monotone** and **almost-pure**, then it is uniquely characterized by the support level \(\rho_i > 0\) of some agent \(i\). This ensures that \(\rho_j = 0\) for \(j < i\) and \(\rho_j = 1\) for \(j > i\). Note that possibly \(\rho_i = 1\).

For each \(i \in \{1, \ldots, n\}\), let \(P_i = \sum_{j=1}^n p_j\) be the probability that the recognized proposer is either agent \(i\) or some agent to her right. Let \(\beta_i = b_{P_i} = \arg\max_{b \in X} u((b - x_l)^{1-P_i} u(b - x_r)^{P_i})\).

**Corollary 1.** Let \(i^* = \min \{i | x_i > \beta_{i+1}\}\). Then, the equilibrium policy is \(b^* = \min \{x_{i^*}, \beta_{i^*}\}\). Moreover, there exists a unique equilibrium in monotone and almost-pure strategies, in which \(\rho_j = 0\) whenever \(j < i^*\), \(\rho_j = 1\) whenever \(j > i^*\), and \(\rho_{i^*} > 0\) with \(\rho_{i^*} < 1\) only if \(x_{i^*} < \beta_{i^*}\).

### 3.3 Limit Equilibria

I now turn my attention to characterizing the limit equilibria. By Proposition 2, we know that as \(\delta \to 1\), the equilibrium proposals of all players converge. Let \(\mu = \lim_{\delta \to 1} y_i\) for each \(i\). To build intuition for the main result, suppose \(\mu \in (x_{i-1}, x_i)\). Then, for \(\delta\) large enough, it must be that \(x_{i-1} < x_r < \mu < x_l < x_i\), so that all players \(j \leq i\) propose \(y_j = x_r\) and all players \(j \geq i + 1\) propose \(y_j = x_l\). Furthermore, we know that \(x_r\) and \(x_l\) are pinned down by the preferences of players \(r\) and \(l\) respectively. We can think of the equilibrium as the result of a bargain between players \(l\) and \(r\), where \(l\) negotiates on behalf of all agents \(j < i\) and \(r\) negotiates on behalf of all agents \(j \geq i\). The main result of this paper is that, roughly speaking, as \(\delta \to 1\), the equilibrium policy converges to the asymmetric Nash Bargaining solution that gives bargaining weights to \(l\) and \(r\) in proportion to the cumulative recognition probabilities of the players in their ‘faction’.

Formally, for \(i = 1, \ldots, n\), let \(P_i = \sum_{j=i}^n p_j\) be the probability that the proposer is weakly to
the right of player $i$. For each $i$, let

$$b_i = \arg \max_{y \in X} u(y - x_l)^{1-P_i} u(y - x_r)^{P_i}$$

denote the $i^{th}$ (asymmetric) Nash Bargaining solution. Since $x_l \leq x_r$, we should expect the solution to be weakly closer to $x_l$ as the bargaining power of agent $l$ increases. It is easy to show that $b_i \leq b_j$ whenever $i > j$, and that this inequality is strict whenever $x_l < x_r$.

**Proposition 4.** Let $i^* = \min \{i \mid x_i \geq b_{i+1}\} \in \{l + 1, \ldots, r\}$. Then the limit equilibrium is characterized by:

- If $x_{i^*} > b_{i^*}$, then $\mu = b_{i^*}$.
- If $x_{i^*} \leq b_{i^*}$, then $\mu = x_{i^*}$.

To make sense of Proposition 4, return to the heuristic argument above. $b_i$ is the solution to the Nash bargaining problem between players $l$ and $r$, where the bargaining strengths of $l$ and $r$, respectively, are $1 - P_l = \sum_{j < i} p_j$ and $P_i = \sum_{j \geq i} p_j$. These bargaining weights were motivated by the idea that, for $\delta$ sufficiently large, all players $j < i$ would choose $y_j = x_r$ and all players $j \geq i$ would propose $y_j = x_l$. Consistency requires $x_{i-1} < b_i < x_i$, otherwise the factions would not be $\{1, \ldots, i - 1\}$ and $\{i, \ldots, n\}$. Now, if $i < i^*$, then by construction, $x_{i-1} \leq x_i < b_{i+1} \leq b_i$, which is inconsistent with the above logic. Similarly, if $i > i^*$, then by construction $b_i \leq b_{i-1} \leq x_{i-1} \leq x_i$, which is also inconsistent with the above logic. This explains the focus on $i = i^*$.

Identifying player $i^*$ is necessary, but not sufficient, to characterize the limit equilibrium. From here, there are two possibilities: (i) $x_{i^*-1} < b_{i^*} < x_{i^*}$ and (ii) $x_{i^*-1} \leq x_{i^*} \leq b_{i^*}$ (with at least one inequality strict). The former case is consistent with our story, and implies that $\mu = b_{i^*}$. In the latter case, there is a problem. If we believe player $i^*$ is in $r$’s faction, then the Nash bargaining solution selects a limit policy $\mu > x_{i^*}$, that would cause player $i^*$ to
want to be in $l$'s faction. Similarly, if we believed player $i^*$ to be in $l$'s faction, then the Nash bargaining solution would choose limit policy $\mu < x_{i^*}$ which would cause player $i^*$ to be in $r$'s camp. Player $i^*$ is pivotal. The only possibility is that the limit equilibrium coincides with his ideal policy, $x_{i^*}$.

Proposition 4 gives a simple and tractable characterization of the limit equilibrium. The equilibria of bargaining games are typically characterized as the fixed points of some mapping. In general, these are difficult to compute, making their use in applied models cumbersome. Proposition 4 reduces the characterization of the limit equilibrium to a simple decision problem — the modeler need not solve a complicated game nor find the fixed points to some complicated system. The limit equilibria follow immediately from the first order conditions to some decision problem.

**Example 1.** Let $n = 7$ and $q = 6$, so that decision making requires one less than unanimity. Then $l = 2$ and $r = 6$. Suppose $u_i(y) = 1 - |y - x_i|$, with $x_1 \leq ... \leq x_7$, and normalize $x_2 = 0$ and $x_6 = 1$. Finally, suppose $p_i = \frac{1}{7}$ for each $i$, so that each player is recognized to propose with equal probability. It follows that:

$$b_i = P_i x_l + (1 - P_i) x_r + (2P_i - 1)$$
so that \( b_i = \frac{8-i}{7} \), for each \( i \). Hence, we have:

\[
\mu = \begin{cases} 
\frac{5}{7} & x_3 > \frac{5}{7} \\
\frac{4}{7} & x_3 \leq x_4 \leq \frac{5}{7} \\
\frac{3}{7} & x_4 < \frac{3}{7} < x_5 \\
\frac{2}{7} & x_5 \leq x_5 \leq \frac{3}{7} \\
\frac{2}{7} & x_5 < \frac{2}{7} 
\end{cases}
\]

The player 4 is the median, and indeed when \( x_4 \in \left[\frac{3}{7}, \frac{4}{7}\right] \), the limit equilibrium selects the median player’s ideal policy. However, under alternative arrangement of the players’ ideal policies, the limit equilibrium policy may also coincide with the ideal policies of players 3 and 5. By contrast, there is no arrangement of ideal policies under which the limit policy coincides with the ideal policies of players 1, 2, 6 or 7. In the remaining cases, the limit policy coincides with the Nash Bargaining solutions.

The characterization of the limit equilibrium as the asymmetric Nash bargaining solution to a bilateral bargaining problem between the left and right decisive players admits the following nice interpretation. Although the bargaining game is played by \( n \), each with potentially distinct preferences, as players become patient, they separate into two factions and delegate the bargaining to a single representative agent. The composition of the equilibrium factions is endogenous to the model, and depends on the players’ recognition probabilities and the spatial alignment of their ideal policies. The bargaining framework, thus provides micro-foundations for the separation of diverse agents into cohesive coalitions or voting blocs, which are represented by agents with non-median preferences. Such behavior is a commonly observed feature in legislature as well as on the U.S. Supreme Court. (** Cite **)
My characterization of the limit equilibrium coincides with that of Predtetchinski (2011). He shows that the limit equilibrium is the generalized root of the characteristic function defined by:

\[ \xi(x) = \frac{u'_l(x)}{u_l(x)} \sum_{j:x_j \leq x} p_j + \frac{u'_r(x)}{u_r(x)} \sum_{j:x_j > x} p_j \]

Predtetchinski shows that this function is strictly decreasing and has discontinuities. Suppose the generalized root of this function is \( x^* \). There are two possibilities: either (i) \( \xi(x^*) = 0 \) or (ii) \( \lim_{x \uparrow x^*} \xi(x) > 0 > \lim_{x \downarrow x^*} \xi(x) \). In the first case, \( x^* \) is an actual root of the characteristic function. In the second case, \( \xi \) has a discontinuity at \( x^* \) such that it is positive below \( x^* \) and negative above. By inspection, the first case corresponds to the bilateral asymmetric Nash bargaining solution between agents \( l \) and \( r \), when the bargaining weights on \( l \) and \( r \) are \( \sum_{j:x_j \leq x} p_j \) and \( \sum_{j:x_j > x} p_j \), which is precisely \( b_i \) if \( x^* \in (x_{i-1}, x_i) \). Furthermore, since \( u \) is continuously differentiable, the only points of discontinuity of \( \xi(x) \) occur at the ideal policies \( x_j \) of agents who are recognized with positive probability. (At these points, there is a discontinuous jump in the cumulative probabilities \( \sum_{j:x_j > x} p_j \) and \( \sum_{j:x_j \leq x} p_j \).) Hence, the second case, where \( x^* \) is a generalized root of \( \xi \) must correspond to the ideal policies of some agent who is recognized with positive probability. Moreover, since the limit policy must be in the core \([x_l, x_r]\), the candidate pivotal agents must have ideal policy strictly within the core.

Predtetchinski (2011) (on p. 536), limiting attention to preferences that are symmetric and differentiable, nevertheless claims that the limit policy is generically characterized by neither the asymmetric Nash bargaining solution, nor the median agent's ideal policy. The second part of the claim is partially true — if the limit policy may coincide with the ideal policy of any agent whose ideal policy lies within the core, which includes, but is not necessarily restricted to be, the median player. The reason why Predtetchinski and I differ in the first

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5In fact, his paper provides a more general definition of the characteristic function, for a world where preferences are not necessarily differentiable or symmetric. The expression below gives the special case of the characteristic function when those properties hold.
part is that he focuses on Nash bargaining problems of the form: $\max \Pi_j u_j(x)^{p_j}$ — i.e. of the form considered by Laruelle and Valenciano (2008) and Miyakawa (2008). As I show, the relevant bargaining problem is not between all $n$ players, but simply the bilateral problem between the left and right decisive players alone.

## 3.4 Properties of Limit Equilibria

I now consider some salient properties of limit equilibria.

**Lemma 1.** Suppose $n$ is odd, and $q = \frac{n+1}{2}$, so that agreement requires a simple majority. Then the left and right decisive agents both coincide with the median agent ($l = \frac{n+1}{2} = r$), and the limit policy coincides with the median agent’s ideal ($\mu = x_{\frac{n+1}{2}}$).

This is consistent with Cho and Duggan (2009) and Predtetchinski (2011). If $n$ is odd and agreement requires a simple majority, then the core is simply the ideal policy of the median voter. It follows immediately that this is the policy that is selected in the limit equilibrium.

Next, suppose that there is a pure super-majority rule. If so, there exists some player $i$ whose index satisfies $l < i < r$. (To see this, note that pure super-majority implies $r = q > \frac{n}{2} + 1$ and so $l = n - q + 1 < \frac{n}{2}$ and so $r - l > 1$.) If $x_l = x_r$, then $x_l = x_i = x_r$ for every $i \in \{l+1, ..., r-1\}$. The core remains a singleton that coincides with the median player’s ideal policy and Lemma 1 obtains. The situation becomes more interesting if $x_l < x_r$, so that the core is an interval. From herein, I assume that $x_l < x_r$.

**Lemma 2.** Under pure super-majority rule, there exist recognition probabilities $p_1, ..., p_n$ and an arrangement of ideal policies $x_1 \leq ... \leq x_n$ with $x_l < x_r$ s.t. player $i$ is pivotal for each $i \in \{l+1, ..., r-1\}$. Moreover, if player $i$ is pivotal, then $x_i \in (x_l, x_r)$ and, generically, $p_i > 0$. 

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Lemma 2 states that, under a pure super-majority rule, every agent between the left and right decisive agents is pivotal under some arrangement of ideal policies. When \( n \) is odd, this will include the median player. But it will also include other agents. We see this in example 1, above. In that example, there are 7 agents and the majority requirement is one less than unanimity. The left and right decisive agents are players 2 and 6, respectively, and the median agent is player 4. The lemma states that, under some arrangement of ideal policies and recognition probabilities, each of players 3, 4 and 5 will be pivotal. The example describes various arrangements of ideal policies under which each of these outcomes will obtain, assuming uniform recognition probabilities. Lemma 2 has two important implications. First, players whose ideal policies lie outside the core cannot be pivotal. Second, players not recognized with positive probability cannot generically be pivotal.\(^6\)

Lemma 2 speaks to the pivotality of an agent whose ideal policy lies in the core. The next two lemmas demonstrate more generically which core policies may be selected in the limit equilibrium.

**Lemma 3.** The limit equilibrium selects a policy in the core. More specifically, \( \mu \in [b_r, b_l] \subset [x_l, x_r] \). Furthermore, the set inclusion is strict whenever \( \sum_{j=l+1}^{r-1} p_j > 0 \).

Lemma 3 shows that the limit policy will always be contained in the core. This is consistent with Theorem 5 in Banks and Duggan (2000). Lemma 3 additionally shows that if the players between the left and right decisive agents are collectively recognized with positive probability, then the limit equilibrium will be contained within a strict subset of the core. Policies in the core that are too close to either the left or right decisive agent’s ideal policies, respectively, cannot be sustained as the equilibrium of a bargaining game with patient players. This result is particularly salient in much of the applied literature, where players are assumed to

\[^6\text{If } p_i = 0, \text{ player } i \text{ will not be generically pivotal, in the sense that, following a small perturbation in probabilities and ideal policies, player } i \text{ will not remain pivotal. However, it is possible to find a vector of recognition probabilities for which the Nash bargaining solution coincides with } x_i. \text{ In this sense, the limit policy will select player } i \text{'s ideal policy -- but this is a knife-edge situation.}\]
be equally recognized to make proposals. If so, Lemma 3 implies that the limit policy will likely be ‘moderate’, in the sense of being towards the middle of the core, rather than at the extremes.

**Lemma 4.** Let $x_1, \ldots, x_n$ be an arbitrary arrangement of ideal policies. For each policy $y \in [x_l, x_r]$ in the core, there exists an assignment of recognition probabilities $p_1, \ldots, p_n$ s.t. the limit policy is $y$. Moreover, if there exists $p > 0$ s.t. $p_i \geq p$ for each $i$, then the set of limit policies that can be implemented is a strict subset of $(x_l, x_r)$.

Lemma 4 shows that the limit equilibrium may select any policy in the core given an appropriate assignment of recognition probabilities. For example, to select $x_l$ it suffices to set $p_1 = 1$ and $p_j = 0$ for all $j > 1$, and similarly, to select $x_r$ it suffices to set $p_n = 1$ and $p_j = 0$ for all $j < n$. Furthermore, any policy in the interior of the core can be selected, and this is true regardless of the alignment of ideal policies. Lemma 4 shows that the salient feature in generating limit policies is recognition probabilities rather than the ideal policies of agents. Any limit policy that can be generated using a carefully chosen alignment of ideal policies can also be generated assuming ideal policies are randomly distributed and given appropriately chosen recognition probabilities. However, as the next remark shows, the reverse is not true — fixing a vector of recognition probabilities, there is no guarantee that every policy in the core can be generated given an appropriate alignment of ideal policies.

**Remark 1.** The reverse of Lemma 4 is not true. Given $p_1, \ldots, p_n$, not every policy in the core can be generically implemented given an appropriate alignment of ideal policies $x_1, \ldots, x_n$.

Remark 1 is an immediate consequence of the second part of Lemma 3.

### 3.5 Distribution of Limit Equilibria

[This section contains conjectures and is very preliminary.]
I now turn my attention to the distribution of limit equilibria under different super-majority rules. The basic insight is that, as the super-majority rule becomes more demanding, the limit policy selected in the bargaining game becomes more ‘moderate’. This is made clear in the following example.

**Example 2.** Let $n = 5$, and suppose players are recognized with equal probability, $p_i = \frac{1}{5}$. Preferences are given by $u_i(y) = 1 - |y - x_i|$, with $0 = x_1 \leq \ldots \leq x_5 = 1$. The limit equilibria under different super-majority requirements $q \in \{3, 4, 5\}$ are given by:

- If $q = 3$, then $l = r = 3$, and so $\mu = x_3$.

- If $q = 4$, then $l = 2$ and $r = 4$. Then $b_i = (1 - \frac{i}{5}) x_2 + \frac{i}{5} x_4 + (1 - \frac{2i}{5}) = (1 + x_2) + \frac{i}{5} (x_4 - x_2 - 2)$, and:

$$
\mu = \begin{cases} 
\frac{3}{5} x_2 + \frac{2}{5} x_4 + \frac{1}{5} & x_3 > \frac{3}{5} x_2 + \frac{2}{5} x_4 + \frac{1}{5} \\
\frac{3}{5} x_2 + \frac{3}{5} x_4 - \frac{1}{5} & \frac{3}{5} x_2 + \frac{3}{5} x_4 - \frac{1}{5} \leq x_3 \leq \frac{3}{5} x_2 + \frac{2}{5} x_4 + \frac{1}{5} \\
\frac{2}{5} x_2 + \frac{3}{5} x_4 - \frac{1}{5} & x_3 < \frac{2}{5} x_2 + \frac{3}{5} x_4 - \frac{1}{5} 
\end{cases}
$$

- If $q = 5$, then $l = 1$ and $r = 5$. Then $b_i = \frac{i}{5} + (1 - \frac{2i}{5}) = 1 - \frac{i}{5}$, and:

$$
\mu = \begin{cases} 
\frac{4}{5} & x_2 > \frac{4}{5} \\
\frac{3}{5} & \frac{3}{5} \leq x_2 \leq \frac{4}{5} \\
\frac{3}{5} & x_2 < \frac{3}{5} < x_3 \\
\frac{2}{5} & \frac{2}{5} \leq x_3 \leq \frac{3}{5} \\
\frac{2}{5} & x_3 < \frac{2}{5} < x_4 \\
\frac{1}{5} & \frac{1}{5} \leq x_4 \leq \frac{2}{5} \\
\frac{1}{5} & x_4 < \frac{1}{5}
\end{cases}
$$
Example 2 characterizes the limit equilibria under different super-majority rules, as the alignment of ideal policies varies. Under simple majority rule, the limit equilibrium is uniquely the median player’s ideal policy, and this will be true even if the median agent’s ideal policy is relatively ‘extreme’. Under unanimity \((q = 5)\) and almost-unanimity \((q = 4)\), the median player continues to be pivotal when her ideal policy is ‘moderate’. However, as the median player’s ideal policy becomes more extreme, these regimes substitute away from the median player’s ideal policy to some less extreme policy, which may either be the asymmetric Nash bargaining solution of the bilateral bargaining game between the left and right decisive agents, or the ideal policy of some less extreme agent.

To analyze this issue more formally, suppose there are \(n\) players. Let \(Z_i\) be the ideal policy of agent \(i\), for \(i = 1, \ldots, n\), and suppose each agent’s ideal policy is a draw from an i.i.d distribution \(F_Z\) on \([0, 1]\) which admits a density \(f_Z\). Re-order the agents according to their ideal policies. I.e. let \(X_1 \leq \ldots \leq X_n\), where \(X_1 = \min \{Z_1, \ldots, Z_n\}\), \(X_2 = \min \{Z_1, \ldots, Z_n\} / \{X_1\}, \ldots, X_n = \max \{Z_1, \ldots, Z_n\}\). Then \(X_1, \ldots, X_n\) are the order statistics of \(\{Z_1, \ldots, Z_n\}\). By properties of order statistics, we know that the joint pdf of \(X_1, \ldots, X_n\) is:

\[
f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = n! \prod_{i=1}^{n} f_Z(x_i)
\]

Let \(X = (X_1, \ldots, X_n)\) be a random ordered vector, and let \(p = (p_1, \ldots, p_n)\) be a fixed vector of recognition probabilities, where the agents are ordered by their ideal policies. Then let \(B_i(X, p)\) be the \(i^{th}\) Nash bargaining solution, given ideal policies \(X\) and recognition probabilities \(p\), for \(i = 1, \ldots, n\). Since \(X\) is a random vector, then \(B_1, \ldots, B_n\) are random variables as well. Finally, let \(Y_q(X, p)\) be a random variable that determines the limit policy for each \(q > \frac{n}{2}\), according to Proposition 4. Finally, let \(F_{Y,q}(y)\) be the distribution function for \(Y_q\).

**Conjecture 1.** Suppose \(q < q'\). There exists \(y^* \in (0, 1)\) s.t. \(F_{Y,q}(y) > F_{Y,q'}(y)\) whenever \(y < y^*\) and \(F_{Y,q}(y) < F_{Y,q'}(y)\) whenever \(y > y^*\).
Conjecture 1 states the distribution of limit policies is ‘more dispersed’ when the super-majority rule is less demanding (low \( q \)), than when it more demanding (high \( q \)). This follows from the fact that when \( q' > q \), \( F_{Y,q'} \) crosses \( F_{Y,q} \) exactly once, and from below. This implies that \( F_{Y,q'} \) puts less weight on extreme outcomes, and more weight on moderate outcomes. We can see this in the following example:

**Example 3.** Let \( n = 3 \), and suppose players are recognized with equal probability, \( p_i = \frac{1}{3} \). Preferences are given by \( u_i(y) = 1 - |y - x_i| \). Let \( Z_i \sim F_Z \) on \([0, 1]\), where \( f_Z = 12 \left(x - \frac{1}{2}\right)^2 \).

In the diagram below, the red, yellow and purple lines represent the marginal distributions of the ideal policies of agents 1, 2 and 3, respectively. The yellow line, thus, coincides with the ideal policy of the median agent, which is the distribution of the limit policy when \( q = 2 \) (simple majority). The blue line represents the distribution of the limit policy under unanimity \( q = 3 \). Clearly the limit policy is more dispersed under simple majority.

**Conjecture 2.** Suppose \( F_Z \) is symmetric about its mean. If \( q' > q \), then \( Y_q \) is a mean-preserving spread of \( Y_{q'} \).
4 Extensions

4.1 Reversion to Status Quo

[To be written]

5 Conclusion

[To be written]

6 Appendix

Proof of Proposition 3. First, take the case when \( x_l = x_r \). Then, for every \( \rho \), the unique maximizer of of the Nash Bargaining problem is \( b = x_l = x_r \). (This follows, since the Nash Bargaining problem maximizes \( u(b - x_l)^{1-\rho} u(b - x_r)^\rho = u(b - x_l) \), and this is uniquely maximized when \( b = x_l \).

Next, suppose \( x_l < x_r \). Then since \( u \) is concave, the Nash Bargaining problem \( \max_b u(b - x_l)^{1-\rho} u(b - x_r)^\rho \) has a unique maximizer \( b_\rho \). Moreover, \( b_\rho \) is strictly increasing in \( \rho \).

Let \((\rho_1,\ldots,\rho_n)\) be an equilibrium vector of support levels, and let \( b_\rho \) be the corresponding Nash Bargaining solution. Then, if \( x_i < b_\rho \), it must be that \( \rho_i = 0 \), or else agent \( i \) could choose some \( \rho'_i < \rho_i \) and move the policy closer to her ideal. Similarly, if \( x_i > b_\rho \) then \( \rho_i = 1 \).

Now, let \((\rho_1,\ldots,\rho_n)\) and \((\rho'_1,\ldots,\rho'_n)\) be equilibria and suppose the implied policies \( b_\rho \) and \( b_{\rho'} \) are distinct. WLOG, suppose \( b_\rho < b_{\rho'} \). Let \( L = \{ i \mid \rho_i = 0 \} \) and \( L' = \{ i \mid \rho'_i = 0 \} \) be the sets of agents allocating all of their support to the left faction, in each equilibrium,
and define $R$ and $R'$ similarly. Clearly, if $\rho_i < 1$, then $\rho'_i = 0$. (To see this, note that, if $\rho_i < 1$, then $x_i \leq b_\rho < b'_{\rho'}$, which by the previous discussion requires $\rho'_i = 0$.) Hence, $\rho = \sum_i \rho_i \geq \sum_i \rho'_i = \rho'$. Then, by the monotonicity property, $b_\rho \geq b'_{\rho'}$. But this contradicts the assumption that $b_\rho < b'_{\rho'}$. Hence, there cannot be multiple equilibria with distinct equilibrium policies.

Let $b^*$ be the unique policy induced by any equilibrium. It follows $\rho_i = 0$ whenever $x_i < b^*$ and $\rho_i = 1$ whenever $x_i > b^*$.

Proof of Corollary 1. S

Proof of Proposition 4. Take any $i \in \{1, \ldots, n\}$, and suppose $\mu \in (x_{i-1}, x_i)$. Then, by Proposition 2, there exists $\bar{\delta} < 1$ s.t. for $\delta > \bar{\delta}$, $x_{i-1} < x_r(\delta) < x_l(\delta) < x_i$. (For clarity, I make explicit the dependence of $x_r$ and $x_l$ on $\delta$.) Then, by Proposition 1, all players $j \in \{1, \ldots, i-1\}$ will propose $x_r$ and all players $j \in \{i, \ldots, n\}$ will propose $x_l$. Again by Proposition 1, this implies that:

$$u(x_r - x_r) = \delta [(1 - P_i) u(x_r - x_r) + P_i u(x_l - x_r)]$$

(1)

$$u(x_l - x_l) = \delta [(1 - P_i) u(x_r - x_l) + P_i u(x_l - x_l)]$$

(2)

where $P_i = \sum_{j \geq i} p_j$. By the implicit function theorem, this system of equations pins down $x_r$ and $x_l$ in terms of the model parameters.

Now, let $E[y] = (1 - P_i) x_r + P_i x_l$. Note, by construction, that $x_r < E[y] < x_l$. Then $x_l - E[y] = \frac{1-P_i}{P_i} (E[y] - x_r)$. We effect the following change of variables: Let $\varepsilon = E[y] - x_r$. Then, we have: $x_r = E[y] - \varepsilon$ and $x_l = E[y] + \frac{1-P_i}{P_i} \varepsilon$. Equations (1) and (2) become:

$$(1 - \delta (1 - P_i)) u(E[y] - \varepsilon - x_r) = \delta P_i u \left( E[y] + \frac{1-P_i}{P_i} \varepsilon - x_r \right)$$

(3)

$$(1 - \delta P_i) u \left( E[y] + \frac{1-P_i}{P_i} \varepsilon - x_l \right) = \delta (1 - P_i) u(E[y] - \varepsilon - x_l)$$

(4)
By the implicit function theorem, and since \( u \) is continuously differentiable, we have:

\[
\begin{bmatrix}
(1 - \delta (1 - P_i)) u'_r(x_r) - \delta P_i u'_r(x_l) & -(1 - \delta (1 - P_i)) u'_r(x_r) - \delta (1 - P_i) u'_r(x_l) \\
(1 - \delta P_i) u'_l(x_l) - \delta (1 - P_i) u'_l(x_r) & (1 - P_i) u'_l(x_l) + \delta (1 - P_i) u'_l(x_r)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial E[y]}{\partial \delta} \\
\frac{\partial E[x]}{\partial \delta}
\end{bmatrix} = \begin{bmatrix}
(1 - P_i) P_i u'_l(x_l) \\
(1 - P_i) P_i u'_r(x_r)
\end{bmatrix}
\]

Taking limits as \( \delta \to 1 \), we have:

\[
\begin{bmatrix}
0 & -u'(\mu - x_r) \\
0 & \frac{1 - P_i}{P_i} u'(\mu - x_l)
\end{bmatrix}
\begin{bmatrix}
\lim_{\delta \to 1} \frac{\partial E[y]}{\partial \delta} \\
\lim_{\delta \to 1} \frac{\partial E[x]}{\partial \delta}
\end{bmatrix} = \begin{bmatrix}
u(\mu - x_r) \\
\nu(\mu - x_l)
\end{bmatrix}
\]

These imply that:

\[
\lim_{\delta \to 1} \frac{\partial \nu}{\partial \delta} = -\frac{\nu(\mu - x_r)}{u'(\mu - x_r)} = \frac{P_i}{1 - P_i} \frac{\nu(\mu - x_l)}{u'(\mu - x_l)}
\]

The second equality provides an equation that uniquely defines the limit equilibrium. Let \( b_i \) be the solution to this equation.

Next, we note that equation defining \( \mu_i \) coincides with the first order condition of the \( i^{th} \) Nash Bargaining problem. Recall, that problem was: \( \max_{y \in \mathcal{X}} \ u(y - x_l)^{1 - P_i} u(y - x_r)^{P_i} \).

Since utilities are concave, the maximizer must be the solution to the first order condition:

\[
(1 - P_i) \frac{u'(b_l - x_l)}{u(b_l - x_l)} + P_i \frac{u'(b_r - x_r)}{u(b_r - x_r)} = 0.
\]

Re-arranging gives the desired result.

Notice that \( b_i \) is increasing in \( P_i \). (To see this, re-arranging the first order condition to give:

\[
\frac{u'(b_l - x_l)}{u'(b_r - x_r)} \cdot \frac{u(b_l - x_l)}{u(b_r - x_r)} = -\frac{P_i}{1 - P_i}.
\]

We know that \( b \in [x_l, x_r] \). By single-peakedness, we know that \( u(b - x_l) \) is strictly decreasing in \( b \) and \( u(b - x_r) \) is strictly increasing in \( b \), so \( \frac{u(b - x_r)}{u(b - x_l)} \) is strictly decreasing in \( b \). Similarly, by concavity of \( u \), \( u'(b - x_l) \) is decreasing in \( b \) and \( u'(b - x_r) \) is increasing in \( b \), and so \( \frac{u'(b - x_r)}{u'(b - x_l)} \) is weakly decreasing in \( b \). Hence, the left hand term is strictly decreasing in \( b \). The right hand term is also strictly decreasing in \( P_i \). Hence, as \( P_i \) increases, so must \( b_i \). Then, since \( P_i \) is decreasing in \( i \), it follows that \( b_i \) is decreasing is as well.

Since we conjectured \( \mu \in (x_{i-1}, x_i) \), then the limit equilibrium policy coincides with \( i^{th} \) Nash
Bargaining solution provided that \( x_{i-1} < b_i < x_i \). Now, since \( x_i \) is increasing and \( b_i \) is decreasing in \( i \), then by the definition of \( i^* \), \( x_i < b_{i+1} \) for all \( i < i^* \) and \( x_i \geq b_{i+1} \) for all \( i \geq i^* \). Moreover, for \( i < i^* \), \( x_{i-1} \leq x_i < b_{i+1} \leq b_i \), which is inconsistent. Similarly, for \( i > i^* \), \( b_i \leq x_{i-1} \leq x_i \), which is inconsistent. Hence, if \( b_i \in (x_{i-1}, x_i) \), then \( i = i^* \). Note however, that the converse need not be true. Setting \( i = i^* \) gives two possibilities: (i) \( x_{i^*-1} < b_{i^*} < x_{i^*} \), or (ii) \( x_{i^*-1} \leq x_{i^*} \leq b_{i^*} \) (with at least one inequality strict). The former case is equilibrium consistent, and since the equilibrium is unique, we have \( \mu = b_{i^*} \).

Suppose the latter case prevails. It follows that the limit equilibrium is not contained in any of the open intervals \((x_{i-1}, x_i)\), and so \( \mu \in \{x_1, \ldots, x_n\} \). (In fact, since \( x_r \leq x_r \) and \( \overline{x}_l \geq x_l \) for all \( \delta \), and since \( \lim_{\delta \to 1} x_r = \mu = \lim_{\delta \to 1} \overline{x}_l \), then \( x_l \leq \mu \leq x_r \), and so \( \mu \in \{x_l, \ldots, x_r\} \).

Suppose \( \mu = x_i \) for some \( i \in \{l, \ldots, r\} \). ****

**Proof of Lemma 1.** Suppose \( n \) is odd and \( q = \frac{n+1}{2} \). Then \( l = n - q + 1 = q = r \). For notational convenience let \( x_{med} = \frac{x_{n+1}}{2} \) denote the ideal policy of the median player. Notice that every limit equilibrium must be contained in the set \([b_r, b_l]\). (I prove this formally in the proof of Lemma 3.) Since \( l = r = \frac{n-1}{2} \), it follows that \( b_r = b_l = x_{med} \). It follows that \( \mu = x_{med} \).

**Proof of Lemma 2.** Suppose \( x_l < x_i < x_r \). If \( \mu = x_i \), then we need \( x_{i-1} \leq x_i \leq b_i \) with at least one inequality strict, and \( b_{i+1} \leq x_i \). Generically, \( x_i < b_i \), and so \( b_{i+1} \leq x_i < b_i \). Since \( b_i \) is a strictly decreasing in \( P_i \), it suffices to assign \( p_1, \ldots, p_n \) s.t. \( b_i \leq x_i < b_{i-1} \). But if \( p_i = 0 \), then \( b_i = b_{i-1} \), which is contradiction. Hence \( p_i > 0 \).

Next, consider the non-generic case where \( x_{i-1} < x_i = b_i \) and \( b_{i+1} \leq x_i \), so that \( b_{i+1} \leq x_i = b_i \). Then, it is possible to assign \( p_1, \ldots, p_n \) with \( p_i = 0 \) s.t. \( b_{i+1} = x_i = b_i \). However, for any order preserving \( \varepsilon \)-perturbation of probabilities and ideal policies, \( x_i \) will no longer be the limit policy. ****

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Proof of Lemma 3. Let $\mu$ be the limit equilibrium. Let $i^* = \min\{x_i \geq b_{i+1}\}$. Then $\mu = \min \{b_{i^*}, x_{i^*}\}$. Now, by construction, $b_i \in [x_l, x_r]$ for every $i$. Then for every $i \leq l$ s.t. $x_i < x_l$, it must be that $x_i < x_l \leq b_i$ and so $i \neq i^*$. Furthermore, by construction, $x_r \geq b_{r+1}$, and so $i^* \leq r$. Hence $i^* \in \{l, ..., r\}$, and so $\mu \in \{\min \{b_l, x_l\}, ..., \min \{b_r, x_r\}\}$.

Now, $\mu = \min \{x_l, b_{i^*}\} \leq b_{i^*} \leq b_l$ since $i^* \in \{l, ..., r\}$ and $b_i$ is decreasing in $i$. Now, suppose $\mu < b_{i^*}$, then $\mu < b_i$ for all $i \leq r$ since $b_i$ is decreasing. Hence, there is some $i^* \in \{l, ..., r\}$ s.t. $b_{i^*} + 1 \leq x_{i^*} \leq b_{i^*}$ and $\mu = x_{i^*}$. But since $\mu < b_i$ for all $i \in \{l, ..., r\}$, the only possibility is $i^* = r$. Hence $\mu = x_r \geq b_r$, which is a contradiction. Hence $\mu \geq b_r$. Hence $\mu \in [b_r, b_l]$.

Finally, suppose $\mu = x_l$. This requires that $\sum_{j=1}^l p_j = 1$, which ensures that $b_l = ... = b_r = x_l$. If $p_j > 0$ for some $j > l$, then $b_l > x_l$ and so $i^* > l$. Similarly, suppose $\mu = x_r$. This requires that $\sum_{j=r}^n p_j = 1$, which ensures that $b_l = ... = b_r = x_r$. If $p_j > 0$ for some $j > l$, then $b_r < x_r$ and so $\mu < x_r$. Hence, to ensure $\mu \in (x_l, x_r)$ it suffices that $\sum_{j=l+1}^{r-1} p_j > 0$. □

Proof of Lemma 4. If $x_l = x_r$, then the result follows immediately from Lemma 1. Suppose $x_l < x_r$. Fix any alignment of ideal policies $x_1 \leq ... \leq x_n$ s.t. $x_l < x_r$. Take any $y \in [x_l, x_r]$. For each $P \in [0, 1]$, let $b(P) = \arg \max_y u(y - x_l)^P u(y - x_r)^{1-P}$. Since $x_l < x_r$, then $b(P)$ is strictly decreasing in $P$. Then $b(P)$ is invertible. For any $y \in [x_l, x_r]$, let $b^{-1}(y)$ define the $P$ s.t. $b(P) = y$. In particular, $b^{-1}(x_l) = 1$ and $b^{-1}(x_r) = 0$.

Now, let $i^*$ be s.t. $x_{i^*} < y \leq x_{i^*+1}$. Then, by Proposition 4 $\mu = y$ provided that $b_{i^*} = y$ and $x_{i^*} < b_{i^*} \leq x_{i^*+1}$. But, by construction $b_{i^*} = b(P_{i^*})$. It suffices to assign probabilities $p_{i^*}, ..., p_n$ s.t. $\sum_{j=1}^{i^*} p_j = b^{-1}(y)$ and $\sum_{j=i^*}^n p_j = 1 - b^{-1}(y)$.

Next, suppose $p_i > p$ for each $i$, with $p > 0$. We know that if $\mu = b_i$, then $i \in \{l, ..., r - 1\}$. Then $b_l \leq b(lP) < x_r$ and $b_{r-1} \geq b((r-1)P) > x_l$. The set of implementable policies is bounded by $(b((r-1)P), b(lP)) \subset (x_l, x_r)$. □
References


