Virtue of compactness

**Definition:** A first-order sentence \( \psi \) is trivial over finite models if it is eventually constant.

**Fact:** Every nontrivial first-order sentence \( \theta \) has infinite models of both it and its negation.

**Proof:** There are arbitrarily large finite models of both \( \theta \) and \( \neg \theta \). Hence both \( \{ \theta \} \cup \Phi_{\omega} \) and \( \{ \neg \theta \} \cup \Phi_{\omega} \) are finitely satisfiable, so by compactness each have a finite model.

Compactness in the finite

**Definition:** A set of FO sentences is finitely consistent if every finite subset has a model.

**Compactness:** Every finitely consistent set of FO sentences has a model (i.e. is consistent).

\[ (\exists x \forall y) \theta(x, y) \equiv \exists x_1 \ldots x_n \land \{ x \neq x_i : 1 \leq i < j \} \]

\[ \Phi_{\omega} = \{ \exists x : n \geq 1 \} \]

**Failure:** Each finite subset of \( \Phi_{\omega} \) has a finite model, but \( \Phi_{\omega} \) does not have a finite model.

Gaifman graph

**Definition:** The Gaifman graph of a relational \( L \)-structure \( S \) is the simple graph over \(|S|\) with

\[ E = \{(a, b) : a \neq b & S \models R(...a, ...b, ...) \text{ in } L\} \]

**Idea:** pair elements occurring jointly in tuple

**Advantage:** Can refer to graph notions such as distance \( d(a, b) \) and degree in any \( L \)-structure.

Connectivity [Gaifman, Vardi, ’85]

**Theorem:** Connectivity is not FO in the finite.

**Proof:** Take \( G \models \theta \leftrightarrow G \) is connected, \(|G| < \infty\), \( T = \{ \forall x, y \} E(x, y) \rightarrow x \neq y \land E(y, x) \} \) (simple)

\( \forall x \exists \gamma E(x, y) \land (\exists \theta) E(x, y) \) (two-regular)

\( \exists x_1 \ldots x_n \{ x \neq x_i \land E(x_i, x_2) \land ... E(x_2, x_3) \} \) (acyclic)

T is consistent with both \( \theta \) and \( \neg \theta \) (separately). So by compactness we get \( T \not\models \neg \theta \) and \( T \not\models \theta \).

Models of \( T \) are unions of infinite chains, so \( T \) is uncountably categorical. \( \therefore \) \( T \) is complete, \( \geq \).

Neighborhoods

**Definition:** The \( r \)-ball \( B_r(a) = \{ b : d(a, b) \leq r \} \).

The radius \( r \)-neighborhood of \( a \) is a structure:

\[ N_r(a) = (B_r(a), R \cap |B_r(a)|^{arity(R)}, ... a) \text{ for all } R \text{ in } L\]

The component of \( a \) is \( N_r(a) = U \{ N_r(a) : r > 0 \} \).

**Tuples:** Define \( d(a, b) = \min \{ d(a, b) : a \text{ in } a \} \).

Extends \( N_r(a) \) and \( N_{\omega}(a) \) naturally for \( |a| > 0 \).
Isomorphism locality

**Theorem:** [Hella, Libkin, Nurmonen, 1999]
Every first-order L-formula $\theta(x)$ is Gaifman local, i.e. there is a radius $r$ such that for all relational $L$-structures $S$ and tuples $a$ and $b$,

$$N_1(a) \cong N_2(b) \implies S \models \theta[a] \iff \theta[b]$$

**Proof:** follows from Gaifman’s theorem, 1982.

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Internalize isomorphism

**Proof:** $\theta(x)$ not Gaifman local means for each $r$

$$G_r \models \theta[a] \land \lnot \theta[b]$$

where $f: N_1(a) \cong N_2(b)$

Take $T = \{\theta(a), \lnot \theta(b), f: N_1(a) \cong N_2(b) : r \geq 0\}$
T is finitely consistent. By compactness we get

$$(G_1, f) = \theta[a] \land \lnot \theta[b] \quad f: N_1(a) \cong N_2(b)$$

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Threshold locality

**Theorem:** [Fagin, Stockmeyer, Vardi, 1995]
Over degree $d$ bounded structures, every first-order sentence $\phi$ is Hanf threshold local: it has a radius $r$ and threshold $t$ such that for all $N$,

$$\{|a \in |A| : N_1(a) \cong N| \|^t \} \equiv |\{b \in |B| : N_2(b) \cong N|\}$$

**Proof:** inspired by Hanf’s lemma, 1965 ($d, t$).

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Use a model pair

$(G_1, G_2, R), R \subseteq V_1 \times V_2$. Theory $T$ says degree $d$ &:

- $\{V_1, E_1\} \models \phi \land \{V_2, E_2\} \models \lnot \phi$ (substitution)
- $\{R(x, y) : N_1(x) \equiv N_2(y) : r > 0\}$ (since size $\sim d'$)
- $\{\forall x \forall y : R(x, y) \land \forall y \exists x : R(x, y) : t > 0\}$ (1 by 1)

If $\phi$ is not threshold local, then by compactness $T$ has a model pair where $R(u, v) \rightarrow N_1(u) \cong N_2(v)$

The isomorphisms form a finitely branching tree under inclusion (König’s infinity lemma). $[u \sim v]$

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Multiplicity of congruence classes

- Let $[e] = \{e' \in |G| : N_1(e) \cong N_2(e')\}$. $T$ implies pointed components occur equi-numerously.
- Let $[N_1(e)] = \{N \subseteq |G| : N \cong N_1(e)\}$. Show the same for these un-pointed components.

If $m = |[e] \cap N_1(e)| < \infty$ then $|[N_1(e)]| = |[e]| \cdot m$
If $m = \infty, \{d(e, e') : e \sim e' \in N_1(e)\}$ is unbounded, so the type $\{d(c_i, c_j) : n : c_i \sim c_j, i, j \in \omega\}$ is consistent. By saturation (WLOG) $|[N_1(e)]| = \infty$.
Hence $G_1 \equiv G_2$, contradicting $G_1 \not\models \phi$. $G_2 \not\models \lnot \phi$. 