Presentation-invariant definability

Steven Lindell, Haverford College
Scott Weinstein, University of Pennsylvania
Elementary definability

Simple graph:  \(- \subseteq V^2 \quad \forall x, y \in V\)

- \(- (x - x) \land (x - y \rightarrow y - x)\)

Total ordering:  \(< \subseteq D^2 \quad \forall x, y, z \in D\)

- \(- (x < x) \land (x \neq y \rightarrow x < y \lor y < x)\)
- \((x < y < z \rightarrow x < z)\)
Undefinability

**even:** The number of vertices is even.

**connected:** The graph is connected.

**acyclic:** The graph is acyclic.

None of these are elementary over *finite* graphs.

Because first-order logic is *local* (compactness).
Order invariance

Augment each graph with an arbitrary ordering:

\[(G, <)\]

Elementary definability invariant of particular <:

\[(G, <) \models \theta \iff (G, <') \models \theta\]

But: **even, connected, acyclic** \(\notin\) \(\text{FO}(<) \neq \text{FO}\)
Presentation invariance

Expand each graph by a definable relation $R$:

$$(\exists R) (G, R) \models \sigma$$

**Special case**: $\sigma$ depends only on $|G|$ and $R$.

Using $R$, define a graph query $Q$, invariant of $R$:

$$(\forall R (G, R) \models \sigma) \ [ (G, R) \models \theta \iff G \in Q]$$
Examples for $P \subseteq S$

Degree: \hspace{1cm} \text{zero} \hspace{1cm} \text{one} \hspace{1cm} \text{two} \\
\hspace{1cm} \text{isolated} \hspace{1cm} \text{barbell} \hspace{1cm} \text{chain}

\textbf{even}: \text{barbells with at most one isolated point.}

\textbf{parity}: \text{barbells where both ends are in } P.

\textbf{majority}: \text{barbells with ends in } P \text{ and } \neg P.

\textbf{Fact}: \text{Distance is not bounded-degree invariant.}
Graph traversals

An ordering of its components, each with the property that every initial segment is connected:

\[
\begin{align*}
[ & \ldots & ] & \ldots [x, y] & \ldots [ & \ldots & ] \\
(\forall v)(\exists x)(\exists y)[x \leq v \leq y](\forall z)(x \leq z \leq y) \\
\{ (\forall w - z)[x \leq w \leq y] \} & \land \{ z \neq x \rightarrow (\exists w - z)[w < z] \}
\end{align*}
\]
Traversals invariance

**Connected**: consists of one component interval

**Acyclic**: no node with two prior neighbors

**Reachable**: both nodes are in same component

- Can use breadth-first and depth-first traversals
- Can also define biconnected and bipartite