A normal form for first-order logic over physically feasible data structures

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Logic and Databases workshop

Questions

1. What is the simplest kind of vocabulary retaining the full expressive power of first-order logic?

2. What (infinite) classes of finite data structures are physically realizable (i.e. scale feasibly)?

- **Simplify**: find a "natural" more elegant approach which does not use actual relations in the classical sense – just predicates and functions!

Singulary logic

Vocabulary: one-place symbols (no commas allowed)

$$\mathbf{S} = \langle S, P_1, \ldots P_m, f_1, \ldots f_k \rangle$$

Monadic predicates: $P(\_)$ $P \subseteq S$ *unary* relation
Monadic functions: $f(\_)$ $f\colon S \to S$ *unary* mapping

Atomic forms: $P(F(x))$ $F = f \circ \ldots$
involving <u>equality</u>: $F(y) = G(z)$ [not a true relation]

Transformations

<u>*L*-structure of arity *k*</u>   <u>singulary *L**-structure</u>
$\mathbf{S} = \langle S, R_1, \ldots, R_m \rangle$ $\mathbf{S}^* = \langle S^*, P_1, \ldots, P_m, f_1, \ldots, f_k \rangle$
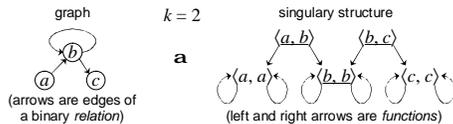
a *k-ary* <u>fact</u>
$R_i(s_1, \ldots s_j \ldots, s_k)$
where $s_j \in S$

$\mathbf{a}$

$\langle s_1, \ldots s_j \ldots, s_k \rangle \in P_i$
$\downarrow f_j$
$\langle s_j, \ldots, s_j \rangle \in S^k$

In a sense, it preserves the *true size* of $\mathbf{S}$ (as a database):
$$|\mathbf{S}| = |S| + |R_1| + \ldots + |R_m| \sim |\mathbf{S}^*| = O(|S^*|).$$

Theorem: For each first-order *L*-sentence θ there is a first-order *L**-sentence θ^* such that $\mathbf{S} \models \theta$ iff $\mathbf{S}^* \models \theta^*$.

Bounded-degree classes

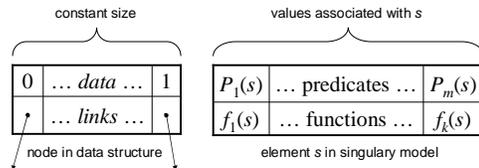out-degree = k, in-degree is preserved (up to a constant)

graph $k = 2$ singulary structure

(arrows are edges of a binary *relation*)   (left and right arrows are *functions*)

Definition: The degree of $\mathbf{S}$ is that of its *Gaifman graph*, relating a and b iff they appear jointly in $\mathbf{S}$.
Theorem: A class of structures is of bounded degree iff the corresponding class of singulary structures is also.

Models for data structures

Information is stored inside nodes of fixed width.
- The number of nodes determines *size* of structure.
- The links between nodes determines its *shape*.

constant size   values associated with *s*

0	… data …	1		$P_1(s)$	… predicates …	$P_m(s)$
	… links …			$f_1(s)$	… functions …	$f_k(s)$

node in data structure   element *s* in singulary model

Physical feasibility

Assume actual implementations use resources:

• *matter* (quantized) occupies *space* at limited density

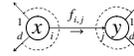Total amount of resources $\propto$ size (no. of nodes).

Nodes are uniform (no *overcrowding* allowed):
$O(1)$ *bits* (mass) and $O(1)$ *connections* (space)

Unbounded in-degree is physically untenable, even though no bound on lengths of connections.
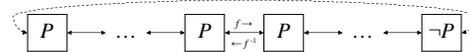
• <u>Conclusion</u>: feasible structures have reversible links

Reversible **a** bijections

Construct partial injections $f_{i,j}^{-1} = f_{j,i}$ (where defined):

Convert (finite) *f*-chains to cycles using *wrap-around*:

Removes *nils* – results in an *bijective* singular model:

$$\langle D, P_1, \ldots, P_m, f_1, \ldots, f_k, f_1^{-1}, \ldots, f_k^{-1} \rangle$$

Normal form result

Definition: For each $n \geq 1$, the *numerical quantifier* $\exists^n x \ldots$ means "there are (at least) n x's such that …".

Theorem: Any formula $\theta(x_1, \ldots x_l)$ in bijective singular logic can be rewritten as a Boolean combination of:

formulas:	$\alpha(x_1, \ldots x_l)$	where α is atomic,
sentences:	$\exists^n x\ \beta(x)$	β is quantifier-free

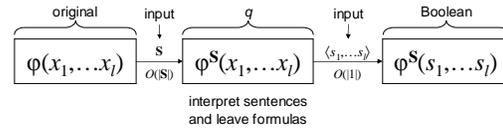**Proof**: syntactic – the key idea is to solve equations:

$$F(x) = G(y) \quad \textbf{a} \quad x = F^{-1} \circ G(y)$$

Linear-time evaluations

Given: a first-order formula $\varphi(x_1, \ldots, x_l)$.

Problem: to compute $\{\langle s_1, \ldots, s_l \rangle : \mathbf{S} \models \varphi(s_1, \ldots, s_l)\}$.

Solution: Calculate a *partial interpretation* query $q = \varphi^{\mathbf{S}}(x_1, \ldots, x_l)$ using time linear in $|\mathbf{S}|$, so that $\varphi^{\mathbf{S}}(s_1, \ldots, s_l)$ can be computed in constant time (i.e. q demonstrates an immediate knowledge of the answer). The <u>method</u>:

Answers

Singular vocabularies are the simplest kind of signature which retain the full power of FOL.

– Any structure can be mapped to a singular one, in which size and degree are preserved uniformly.

Data structures are really just singular models.

– Physically implementable ones are reversibly-linked, in which every function can be made bijective.

Elementary definability over asymptotically scalable classes is much simpler than it appears.

– Using numerical quantifiers, no nesting is required, and all formulas can be evaluated in linear-time.

References

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[Immerman] *Descriptive Complexity*, 1999.
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A full paper is available from my web page.