HAVERFORD

A normal form for first-order logic over physically feasible data structures

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Questions

1. What is the simplest kind of vocabulary retaining the full expressive power of first-order logic?

2. What (infinite) classes of finite data structures are physically realizable (i.e. scale feasibly)?

- **Simplify**: find a "natural" more elegant approach which does not use actual relations in the classical sense – just predicates and functions!

Singulary logic

Vocabulary: one-place symbols (no commas allowed)

$$\mathbf{S} = \langle S, P_1, \ldots P_m, f_1, \ldots f_k \rangle$$

Monadic predicates:	$P(\_)$	$P \subseteq S$	*unary* relation
Monadic functions:	$f(\_)$	$f\colon S \to S$	*unary* mapping

Atomic forms: $P(F(x))$ $\quad F = f \circ \ldots$
involving <u>equality</u>: $F(y) = G(z)$ [not a true relation]

Transformations

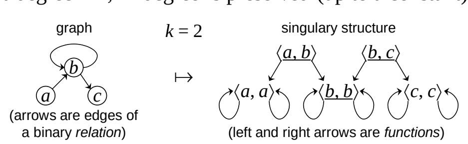
<u>*L*-structure of arity *k*</u>	<u>singulary *L**-structure</u>
$\mathbf{S} = \langle S, R_1, \ldots, R_m \rangle$	$\mathbf{S^*} = \langle S^*, P_1, \ldots, P_m, f_1, \ldots, f_k \rangle$

a *k-ary* <u>fact</u> $R_i(s_1, \ldots s_j \ldots, s_k)$ where $s_j \in S$ $\mapsto$ $\langle s_1, \ldots s_j \ldots, s_k \rangle \in P_i$, $\downarrow f_j$, $\langle s_j, \ldots, s_j \rangle \in S^k$

In a sense, it preserves the *true size* of **S** (as a database):
$$|\mathbf{S}| = |S| + |R_1| + \ldots + |R_m| \sim |\mathbf{S^*}| = O(|S^*|).$$
Theorem: For each first-order *L*-sentence θ there is a first-order *L**-sentence θ* such that $\mathbf{S} \models \theta$ iff $\mathbf{S^*} \models \theta^*$.

Bounded-degree classes

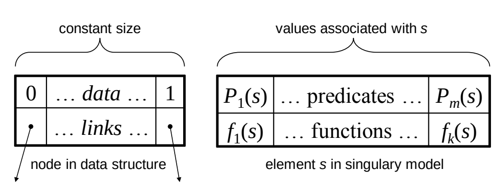
out-degree = *k*, in-degree is preserved (up to a constant)



Definition: The degree of **S** is that of its *Gaifman graph*, relating *a* and *b* iff they appear jointly in **S**.
Theorem: A class of structures is of bounded degree iff the corresponding class of singulary structures is also.

Models for data structures

Information is stored inside nodes of fixed width.
- The number of nodes determines *size* of structure.
- The links between nodes determines its *shape*.



Physical feasibility

Assume actual implementations use resources:

- *matter* (quantized) occupies *space* at limited density

Total amount of resources $\propto$ size (no. of nodes).

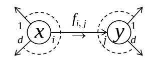
Nodes are uniform (no *overcrowding* allowed): $O(1)$ *bits* (mass) and $O(1)$ *connections* (space)

Unbounded in-degree is physically untenable, even though no bound on lengths of connections.

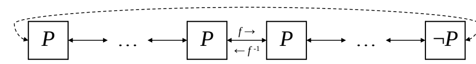
- Conclusion: feasible structures have reversible links

Reversible $\mapsto$ bijections

Construct partial injections $f_{i,j}^{-1} = f_{j,i}$ (where defined):



Convert (finite) f-chains to cycles using *wrap-around*:



Removes *nils* – results in an *bijective* singular model:

$$\langle D, P_1, \ldots, P_m, f_1, \ldots, f_k, f_1^{-1}, \ldots, f_k^{-1}\rangle$$

Normal form result

Definition: For each $n \geq 1$, the *numerical quantifier* $\exists^n x \ldots$ means "there are (at least) n x's such that …".

Theorem: Any formula $\theta(x_1, \ldots x_l)$ in bijective singulary logic can be rewritten as a Boolean combination of:

formulas:	$\alpha(x_1, \ldots x_l)$	where α is atomic,
sentences:	$\exists^n x\, \beta(x)$	β is quantifier-free

Proof: syntactic – the key idea is to solve equations:

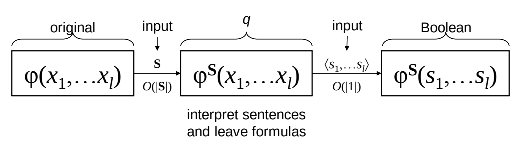
$$F(x) = G(y) \quad \mapsto \quad x = F^{-1} \circ G(y)$$

Linear-time evaluations

Given: a first-order formula $\varphi(x_1, \ldots, x_l)$.

Problem: to compute $\{\langle s_1, \ldots, s_l\rangle : \mathbf{S} \models \varphi(s_1, \ldots, s_l)\}$.

Solution: Calculate a *partial interpretation* query $q = \varphi^{\mathbf{S}}(x_1, \ldots, x_l)$ using time linear in $|\mathbf{S}|$, so that $\varphi^{\mathbf{S}}(s_1, \ldots, s_l)$ can be computed in constant time (i.e. q demonstrates an immediate knowledge of the answer). The method:



Answers

Singulary vocabularies are the simplest kind of signature which retain the full power of FOL.

- Any structure can be mapped to a singulary one, in which size and degree are preserved uniformly.

Data structures are really just singulary models.

- Physically implementable ones are reversibly-linked, in which every function can be made bijective.

Elementary definability over asymptotically scalable classes is much simpler than it appears.

- Using numerical quantifiers, no nesting is required, and all formulas can be evaluated in linear-time.

References

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A full paper is available from my web page.